ORDERED GENERALIZED SEQUENCE SPACES o - MATRIX TRANSFORMATIONS AND ORDER DECOMPOSITIONS

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PRESENTED TO

My Beloved Mother

and

DEDICATED TO

My Father

CERTIFICATE

It is certified that the work contained in the thesis itled "*Ordered Generalized Sequence Spaces, o-Matrix nsformations and Order Decompositions*" by Ms. *Kalika Kaushal*, s been carried out under my supervision and that it has not en submitted elsewhere for any degree or diploma.

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CONTENTS

SYNOPSIS

The theory of Riesz spaces which has its origin in an address of F.Riesz who presented these spaces in the form of function spaces at the international Congress of Mathematicians held at Bologona in 1928, occupies a privileged position in one of the most useful and flourishing branches of mathematics, namely Functional Analysis; indeed, the concept of positiveness in an arbitrary vector space helps overcome the difficulty encountered in dealing with the positive solutions of various problems in non - linear differential and integral equations, as the methods of classical analysis are neither adequate nor simple to yield the solutions of such problems. We may associate the contributions of L.V.Kantorovich, G.Freudenthal, S.Kakutani, G.Birkhoff, B.Z.Vulikh, W.A.J.Luxemburg, A.C. Zaanen, Y.C.Wong, D.H.Fremlin and many others with the development of the theory of ordered vector spaces and Riesz spaces. However, it was M.G.Krein [Zap. Kharkovsk. Matem. Obstich (4),14,227-237] who initiated the work on partially ordered Banach spaces in 1937; though the duality theory of locally convex solid (l.c.s.) Riesz spaces was brought forward by G.T.Roberts in the year 1952 [Proc. Camb. Phil. Soc. Math. Phy. Sci. 48, 533-546]. This theory dealing with the interplay of ordering and topological structures on an ordered vector space or a Riesz space, has now been enriched considerably due to the pioneering work of Krein, Roberts, A.L.Peressini, C.D.Aliprantis, H.H.Schaefer, Y.C.Wong and others; and we have several monographs available now on the subject.

On the other hand, the scalar valued sequence spaces

(SVSS), which have wide applicability in various branches of Functional Analysis, e.g. locally convex spaces, nuclear spaces, summability domains etc., [cf. Lecture Notes, 65, Marcel Dekker, Inc. New York (1980) by P.K.Kamthan and M.Gupta] possesses inherently the order structure induced by the ordering of the real line, when considered over reals. A systematic study of these ordered sequence spaces was made by Peressini and Sherbert in [Math. Ann. 165,(1966), 318-332]. Besides, replacing real line by a vector space, we have a well developed theory of vector valued sequence spaces (VVSS) due to A.Pietsch, D.A.Gregory, R.C.Rosier, De Grande-De Kimpe, P.K.Kamthan, M.Gupta, K.L.N.Rao, J.Patterson etc.. In this dissertation our efforts have been to investigate the results on ordered vector valued sequence spaces (OVVSS) and Riesz spaces by using the tools of the theories of Riesz spaces, l.c.s.Riesz spaces and VVSS. The material of the thesis has been divided into six chapters, a brief scenario of which we present now.

Chapter 1 incorporates some of the basic definitions and results from the theory of ordered vector spaces, Riesz spaces, locally convex spaces, locally solid Riesz spaces and sequence spaces, which serve as a prelude to the subsequent chapters.

Chapter 2 sketches in brief the development of the work carried out in ordered vector spaces and Riesz spaces, locally solid Riesz spaces, vector valued sequence spaces and linear operators on Riesz spaces.

In Chapter 3, we study the impact of ordering and topological properties of a Riesz space and a locally solid Riesz

space X on the (OVVSS) $\Lambda(X)$ defined by the elements of the space X. The main result of the chapter is the identification of the sequential order dual $[\Lambda(X)]^{so}$ of $\Lambda(X)$ with its generalized Köthe dual $\Lambda^{x}(X^{so})$, where $\Lambda(X)$ is an ideal of the space $\Omega(X)$ consisting of all sequences defined by the elements of X. We also introduce ordered vector valued sequence spaces analogous to ℓ^1, ℓ^∞, c, c_o, bv and bv_o with the help of the ordering of an order complete Riesz space, and investigate their generalized Köthe duals.

Chapter 4 is devoted to the study of o-matrix transformations defined on OVVSS $\Lambda(X)$, a concept introduced with the help of order convergence of the Riesz space X. We characterize in this chapter the o-matrix representation of a linear map in terms of its sequential order continuity; besides finding the conditions for matrices of operators to transform certain OVVSS introduced in Chapter 3, into themselves. We also consider the o-precompactness of such transformations.

In Chapter 5, we confine our attention to the duality relationship of a Riesz space X with its order duals, as well as its topological dual in case X is assigned a l.c.s. topology. Infact, we consider polar topologies $\tau_c(X,Y)$ and $\tau_{so}(X,Y)$, defined corresponding to the sets in Y satisfying conditions A_2 and A_1 respectively. where Y is an ideal in the order continuous dual X^c and the sequential order dual X^{so} of X. Further, for a l.c.s.Riesz space X with the topological dual X^*, we also study the topology $T_{op}(X^*,X)$ which is defined with the help of quasi-order precompact sets in X.

The final chapter incorporates results on Riesz spaces possessing countable order decompositions. We show that the

sequential order dual of an order complete Riesz space with countable order decomposition also possesses a countable order decomposition and use the result to derive sufficient conditions for order perfectness of Riesz spaces. Besides, we also introduce associated o-vector valued sequence spaces with the help of a countable order decomposition in a Riesz space and the concept of o-similarity between two countable order decompositions in Riesz spaces which we characterize by making use of associated o-sequence spaces.

CHAPTER 1

PREREQUISITES

1.1 An Overview : This chapter incorporates concepts and results from the theories of ordered vector spaces, Riesz spaces, locally convex spaces, locally solid Riesz spaces and sequence spaces, which are to be used in the subsequent chapters. The results, quoted from various texts, monographs and research papers; for instance, one may refer to [6, 27, 46, 47, 48, 86, 96, 135, 175, 196, 200, 229], are stated without proof. Throughout the sequel, we make use of the following notations

$\mathbb{R}$ = Set of all real numbers

$\mathbb{C}$ = Set of all complex numbers

$\mathbb{N}$ = Set of all natural numbers

$\mathbb{R}_+$ = Set of all positive real numbers

$\mathbb{R}^+ = \mathbb{R}_+ \cup \{0\}$

X = A non trivial vector space over $\mathbb{R}$

X' = Algebraic dual of X

θ = Zero element of X

Besides, the completion of a proof has been marked by the symbol ■.

1.2 Ordered Vector Spaces: Throughout our work in this dissertation, we *deal with the real vector spaces only.* As the basis of our work embodied in this thesis, are ordered vector spaces (OVS) and Riesz spaces, we consider in this chapter various concepts and results from the theories of OVS and Riesz spaces. To begin with, let us recall [175]

Definition 1.2.1: An *ordered vector space* (OVS) is a real vector

space X equipped with a transitive , reflexive and antisymmetric relation $\geq$ satisfying the properties that $x+z \geq y+z$ and $\lambda x \geq \lambda y$ whenever $x \geq y$, for x,y,z in X and λ in $\mathbb{R}^{+}$. In an OVS X, the elements satisfying the conditions that $x \geq \theta$, are known as *positive elements* and the set of such elements is usually known as a *positive cone*.

A set K in a real vector space is said to be a *cone* if it possesses the properties (i) $K + K \subset K$, (ii) $\alpha K \subset K$ for each α in $\mathbb{R}^{+}$, and (iii) $K \cap (-K) = \{\theta\}$. The presence of a cone K in a real vector space X yields the OVS structure of X for the ordering $\geq$ defined as $x \geq y$ if and only if $x-y \in K$. Also the positive cone K of an OVS satisfies the axioms of a cone. Thus there is one to one correspondence between various partial orderings and cones of a real vector space so that it becomes an OVS. Consequently , in the sequel, a positive cone will be referred to a cone only.

A cone K in X *generates* X if $X = K - K$, i.e., X is the linear subspace spanned by K.

Definition 1.2.2: An OVS X is said to be (i) *Archimedean* if $x \leq \theta$ whenever $\lambda x \leq y$ for some $y \in K$ and all $\lambda \in \mathbb{R}_{+}$; (ii) *order complete* (resp.*σ-order complete*) if every directed, majorized subset (resp. countable subset) of X has a supremum in X; and (iii) *order separable* if every subset A of X that has supremum in X contains a countable set A' such that $\sup(A) = \sup(A')$.

In an OVS X, the symbol $[x,y]$ for $x, y \in X$ with $x \leq y$, denotes the set $\{ z \in X : x \leq z \leq y \}$ and is known as an *order interval*. A set $B \subset X$ is said to be *order bounded* if $B \subset [x,y]$ for some $x,y \in X$.

<u>Riesz Spaces</u>: These are special type of OVS defined in

Definition 1.2.3: An OVS X is said to be a *Riesz space* or a *vector lattice* if it contains the supremum $x\vee y$ or infimum $x\wedge y$ of each pair $\{x,y\}$ of its elements.

Unless otherwise specified , X denotes in this subsection a Riesz space . For x in X , the symbols x^+, x^- and $|x|$ have their usual meanings, i.e. $x^+=x\vee\theta$, $x^-=(-x)^+$, and $|x|=x\vee(-x)$. The elementary properties satisfied by the elements of X are listed in [6].

Proposition 1.2.4: For x,y,z in X, we have

(i) $x+y = x\vee y + x\wedge y$;

(ii) $x = x^+ - x^-$; $|x| = x^+ + x^-$; and $x^+\wedge x^- = \theta$;

(iii) $x\vee y = (x-y)^+ + y = (y-x)^+ + x$; and $x\wedge y = x-(x-y)^+ = y-(y-x)^+$;

(iv) $||x|-|y|| \le |x-y|$; and $|x+y| \le |x| + |y|$;

(v) $\lambda(x\vee y) = (\lambda x)\vee(\lambda y)$, $\lambda(x\wedge y) = (\lambda x)\wedge(\lambda y)$ for $\lambda \in \mathbb{R}^+$;and $|\lambda x| = |\lambda||x|$ for all $\lambda \in \mathbb{R}$.

It is clear from the condition (ii) above that the cone in X is always generating. Further, two elements x and y of X are said to be *disjoint* if $|x|\wedge|y| = \theta$, written as $x \perp y$. Thus $x^+$and x^- are disjoint elements for any x in X. The positive elements x and y of a Riesz space X satisfy a property analogous to the real numbers, namely

(1.2.5) $\quad [\theta,x] + [\theta,y] = [\theta,x+y]$.

Also, in order complete (resp. σ -order complete) Riesz spaces which are always Archimedean, the *directed* condition in Definition 1.2.2(ii) is satisfied inherently and so can be relaxed.

Denoting by $x_\alpha \uparrow$ or $x_\alpha\downarrow$ (resp. $x_\alpha\uparrow x$ or $x_\alpha\downarrow x$) an

increasing or a decreasing net $\{ x_\alpha : \alpha \in \Lambda \}$ in an OVS X (resp. increasing or decreasing net with $\sup x_\alpha = x$ or $\inf x_\alpha = x$), let us recall the following [6]:

Definition 1.2.6: A net $\{x_\alpha : \alpha \in \Lambda \}$ in X is said to *order converge* to an element x of X (resp. *order Cauchy*), written as $x_\alpha \xrightarrow{(o)} x$ in X, if there exists a net $\{y_\alpha : \alpha \in \Lambda\}$ such that $y_\alpha \downarrow \theta$ and $|x_\alpha - x| \le y_\alpha$, for each α(resp. $|x_\alpha - x_\beta| \le y_\gamma$, for each $\alpha, \beta \ge \gamma$ in Λ) and in this case, x is said to be the *order limit* of $\{ x_\alpha \}$ which is also written as $x = o\text{-}\lim_\alpha x_\alpha$.

Simple facts concerning the order convergence in X, are listed in

Proposition 1.2.7: (i) If $x_n \xrightarrow{(o)} x$ and $y_n \xrightarrow{(o)} y$ in X, then $x_n + y_n \xrightarrow{(o)} x+y$; $\lambda x_n \xrightarrow{(o)} \lambda x$ for $\lambda \in \mathbb{R}$; and $|x_n| \xrightarrow{(o)} |x|$ in X.

(ii) If $x_n \xrightarrow{(o)} x$ and $y \in X$ is such that $|x_n| \le y$, for each n in $\mathbb{N}$. Then $|x| \le y$.

(iii) If $x_n \uparrow$ (resp. $x_n \downarrow$) in X and $x \in X$ is such that $x = \sup_n x_n$(resp. $\inf_n x_n$), then $x_n \xrightarrow{(o)} x$ in X.

(iv) In an Archimedean Riesz space X, $\lambda_n x \downarrow \theta$ in X whenever $\lambda_n \downarrow 0$ in $\mathbb{R}$ and $x \in K$.

Other types of convergence discussed in our work are contained in

Definition 1.2.8:(i) A net $\{x_\alpha : \alpha \in \Lambda \} \equiv \{x_\alpha\}$ in X is said to *converge relatively uniformly* to an element x of X (resp. *relatively uniformly Cauchy)* if there exists an element a of K, called the *regulator of convergence* , possessing the property that for any $\varepsilon > 0$ there exists $\alpha_o \in \Lambda$ such that $|x_\alpha - x| \le \varepsilon a$, $\forall \alpha \ge \alpha_o$ (resp. $|x_\alpha - x_\beta| \le \varepsilon a$, $\forall \alpha, \beta \ge \alpha_o$); (ii) a sequence $\{x_n\}$ in X is said to *order (*) converge* to an element x of X if every

subsequence of $\{x_n\}$ contains a subsequence that order converges to x.

Several concepts related to the subsets and subspaces of X are mentioned in

Definition 1.2.9: (i) A set B in X is said to be (a) *solid* if $|y|\le|x|$ with x in B yields that $y \in B$; (b) *order closed* (resp. *σ-order closed*) if $x_\alpha \xrightarrow{(o)} x$ (resp. $x_n \xrightarrow{(o)} x$) in X with $\{x_\alpha\}\subset B$ (resp. $\{x_n\}\subset B$), implies that $x \in B$.(ii) *Disjoint complement* $B^{\perp}$ of a set B is defined as the set $\{ y \in X : |y|\wedge|x| = \theta$, for all $x \in B\}$.(iii) A Riesz subspace X_1 of X is said to be (a) *regular* if for any set A in X_1, sup A (resp. inf A) in X_1, if it exists, coincides with the sup A (resp. inf A) in X; (b) an *ideal* of X if it is a solid subspace of X; (c) a *band* of X if it is an order closed ideal of X; and (d) a *projection band* of X if it is a band satisfying $X = X_1 \oplus X_1^{\perp}$.

The smallest solid set containing a set A is known as the *solid hull* of A, which is denoted by s(A) and is the set $\{y \in X : |y| \le |x|$ for some $x \in A \}$. The smallest ideal I_A containing A is known as the *ideal generated by A*, which is given by

$$I_A = \{y \in X: \text{ there exist } x_1, x_2, \ldots, x_n \text{ in X and } \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}^+ \text{ with } |y| \le \sum_{i=1}^{n} \lambda_i |x_i| \}.$$

If $A =\{x\}$, a singleton set in X, I_A is called the *principal ideal* and in addition if it is also order closed , it is called a *principal band* and is denoted by B_x.

Concerning ideals, order closed sets and principal bands, we have the following simple results:

Proposition 1.2.10: (i) Every ideal in a Riesz space X (resp.

order complete Riesz space X) is a regular (resp. order complete) subspace of X.

(ii) A solid set B in X is order closed (resp. σ-order closed) if and only if $\theta \le x_\alpha \uparrow x$ with $\{x_\alpha\}$ (resp. $\theta \le x_n \uparrow x$ with $\{x_n\}$) in B, implies that $x \in B$.

(iii) For an arbitrary positive element y in B_x, there exists a net $\{y_\alpha\} \subset I_{\{x\}}$ such that $\theta \le y_\alpha \uparrow y$ in X.

Characterizing projection bands in X, we have [6].

Proposition 1.2.11: (i) A band B in X is a projection band if and only if $\sup\{y \in B : \theta \le y \le x\}$ exists for all $x \in K$; and

(ii) A principal band B generated by x in X, is a projection band if and only if $\sup\{ y \wedge n|x| : n=1,2\dots \}$ exists for all $y \in K$.

Various properties of a Riesz space are incorporated in [6], [175]

Definition 1.2.12: X is said to have (i) *boundedness property* if a set $B \subset X$ is order bounded whenever $\lambda_n x_n \xrightarrow{(o)} \theta$ in X for any sequence $\{x_n\} \subset B$ and $\lambda_n \downarrow 0$ in $\mathbb{R}$;(ii) *diagonal property* if there is a strictly increasing sequence $\{ m_n : n \ge 1\}$ of positive integers such that $\{ x^{(n,m_n)}\}$ order converges to $x^{(o)}$ for any double sequence $\{x^{(n,m)} : n,m \ge 1 \}$ in X satisfying the conditions that $x^{(n,m)} \xrightarrow{(o)} x^{(n)}$ as $m \longrightarrow \infty$,for each $n \in N$ and $x^{(n)} \xrightarrow{(o)} x^{(o)}$ in X ; (iii) the *projection property* if every band of X is a projection band; (iv) the *principal projection property* if every principal band of X is a projection band ;(v) *sufficiently many projections* if every non trivial band of X contains a non trivial projection band ;and (vi) X is said to be *uniformly complete* (resp. *order Cauchy complete*) if every relatively uniformly

Cauchy sequence(resp. order Cauchy sequence) converges relatively uniformly (resp. order converges) in X with the same regulator of convergence.

Remark : It is clear from the Proposition 1.2.11 that every order complete Riesz space possesses the projection property and every σ-order complete Riesz space has the principal projection property.

For a projection band B in X, we write

$$P_B(x) = \sup \{ y \in B: \theta \leq y \leq x \} \ , \ x \in K \ ;$$

and

$$P_B(x) = P_B(x^+) - P_B(x^-), \qquad x \in X \ ,$$

cf. Definition 1.2.25 below for more details.

Relating order completeness with uniform completeness, order convergence with relative uniform convergence and σ-order completeness with order Cauchy completeness, we have [6], [35], [175]

Proposition 1.2.13: (i) A Riesz space X is order complete if and only if it has projection property and is uniformly complete.

(ii) If X is a σ-order complete Riesz space with diagonal property, then order convergence is equivalent to relative uniform convergence.

(iii) Every σ-order complete Riesz space is order Cauchy complete.

Order Decompositions : The concepts of disjoint elements and sets in a Riesz space X yield a partition of X into particular type of subspaces, which help in representing each element of X in terms of members of these subspaces . Such a partition of X is usually termed as an *order decomposition* . However, before mentioning the

precise definition of this concept, let us recall from [229] a few properties and notions concerning the disjoint elements and disjoint subspaces :

Proposition 1.2.14 : If $x_1, x_2, \ldots, x_n$ are elements of a Riesz space X such that $x_i \perp x_j$ for $i \neq j$, then $|x_1+x_2+\ldots+x_n| = |x_1|+|x_2| +\ldots+|x_n| = \sup\{|x_i| : 1 \leq i \leq n\}$.

Definition 1.2.15 : A collection $\{M_\alpha : \alpha \in \Lambda\}$ of bands in X is said to be (i) *pairwise disjoint* if $M_\alpha \perp M_\beta$ for $\alpha \neq \beta$, i.e., $x \perp y$ for any x in M_α and y in M_β ; and (ii) *complete* if $x = \theta$ whenever $x \perp M_\alpha$ for each α in Λ. (iii) *Union of an arbitrary collection of pairwise disjoint elements* $\{x_\alpha\}$, *denoted* by $\underset{\alpha}{S} x_\alpha$ or $S(x_\alpha : \alpha \in \Lambda)$, is defined by the relation $\underset{\alpha}{S} x_\alpha = \sup x_\alpha^+ - \sup x_\alpha^-$, provided the suprema in the expression exist.

Simple facts concerning the union of disjoint elements are listed in

Proposition 1.2.16: (i) $(\underset{\alpha}{S}\, x_\alpha)^+ = \sup x_\alpha^+$, $(\underset{\alpha}{S}\, x_\alpha)^- = \sup x_\alpha^-$, and $|\underset{\alpha}{S}\, x_\alpha| = \underset{\alpha}{\sup} |x_\alpha|$.

(ii) For a finite number of disjoint elements $\{x_1, x_2, \ldots, x_n\}$, $\sum_{i=1}^{n} x_i = S(x_1, x_2, \ldots, x_n)$.

(iii) For a countable set $\{x_n : n \geq 1\}$ of pairwise disjoint elements, $\underset{n}{S} x_n = \sum_{n=1}^{\infty} x_n = o\text{-}\lim_{n\to\infty} \sum_{i=1}^{n} x_i$.

Remark: For a finite collection of disjoint elements, observe that Proposition 1.2.14 becomes a particular case of Proposition 1.2.16(i) and (ii).

We are now prepared to mention the definition of order decomposition from [229] as follows:

Definition 1.2.17: An *order decomposition* in a Riesz space X is

defined to be a complete system of pairwise disjoint projection bands.

The nomenclature "decomposition" in the above definition is justified by the following result contained in [229].

Theorem 1.2.18: Let $\{M_\alpha : \alpha \in \Lambda\}$ be an order decomposition in an order complete Riesz space X. Then every $x \in X$ has the form $x = \underset{\alpha}{S} x_\alpha$ where $x_\alpha = P_{M_\alpha}(x)$.

<u>o-Linear Operators</u>: As the name suggests , in this subsection we consider results related to linear operators defined on ordered vector spaces . For Riesz spaces X and Y , the *symbol* $\mathcal{L}(X,Y)$ stands for the class of all linear transformations from X into Y. Various types of o-linear mappings from $\mathcal{L}(X,Y)$ are now defined in [6]

Definition 1.2.19: $T \in \mathcal{L}(X,Y)$ is said to be (i) *order bounded* if it maps order bounded subsets of X into order bounded subsets of Y;(ii) *order continuous* (resp. *sequentially order continuous*) if the net $\{T(x_\alpha)\}$(resp. the sequence $\{T(x_n)\}$) order converges to T(x) in Y, whenever $\{x_\alpha\}$(resp. the sequence $\{x_n\}$) order converges to x in X; (iii) *Riesz homomorphism* if $(T(x))\wedge(T(y)) = \theta$ in Y whenever $x \wedge y = \theta$ in X ;(iv) a Riesz homomorphism is said to be (a) *a Riesz isomorphism* if it is one-one,(b) *σ-Riesz homomorphism* if it preserves sequential order convergence, and (c) *normal* if it preserves order convergence. (v) Two Riesz spaces X and Y are said to be *Riesz isomorphic* if there exists a Riesz isomorphism from X onto Y.

The class of all *order bounded linear operators* from X to Y is denoted by $\mathcal{L}^b(X,Y)$. The subspaces of $\mathcal{L}^b(X,Y)$ containing

all *sequentially order continuous* and *order continuous* linear mappings from X to Y are respectively denoted by $\mathcal{L}^{so}(X,Y)$ and $\mathcal{L}^{c}(X,Y)$. For Y = X (resp. $\mathbb{R}$), we write $\mathcal{L}^b(X) \equiv \mathcal{L}^b(X,Y)$ (resp. $X^b \equiv \mathcal{L}^b(X,\mathbb{R})$) , $\mathcal{L}^{so}(X) \equiv \mathcal{L}^{so}(X,Y)$ (resp. $X^{so} \equiv \mathcal{L}^{so}(X,\mathbb{R})$) and $\mathcal{L}^c(X) \equiv \mathcal{L}^c(X,Y)$ (resp. $X^c \equiv \mathcal{L}^c(X,\mathbb{R})$). Also, we use the *notation* X^+ to denote the class of all linear functionals on X, which can be written as the difference of two positive linear functionals on X.

Concerning the order structure of $\mathcal{L}^b(X,Y)$, we have the following well known result contained in [6], [175], [229].

Theorem 1.2.20: Let X and Y be such that Y is order complete. Then

(i) $\mathcal{L}^b(X,Y)$ is an order complete Riesz space with respect to the ordering induced by the cone $\tilde{K} = \{T \in \mathcal{L}^b(X,Y): T(x) \geq \theta , \forall\ x \in K\}$ such that for $T \in \mathcal{L}^b(X,Y)$ and $x \in K$, $|T|(x) = \sup\{|T(y)| : y \in X , |y| \leq x\}$; and for $T \in \tilde{K}$ and $x \in X$, $T(|x|) = \sup\{|S(x)| : S \in \mathcal{L}^b(X,Y), |S| \leq T\}$. Further, for two positive operators T_1, T_2 and $x \in K$, $(T_1 \wedge T_2)(x) = \inf\{ T_1(y) + T_2(z) : y,z \in K, y + z = x\}$.

(ii) $\mathcal{L}^{so}(X,Y)$ and $\mathcal{L}^c(X,Y)$ are bands in $\mathcal{L}^b(X,Y)$.

(iii) for $T \in \tilde{K}$, T is sequentially order continuous if $T(x_n) \downarrow \theta$ in Y whenever $x_n \downarrow \theta$ in X.

In this dissertation our study is confined to those Riesz spaces X for which $\langle X,X^{so}\rangle$ forms a dual pair. Since X^{so} is an order complete Riesz space , we can define the space $(X^{so})^{so}$ and so a notion of order perfectness of a Riesz space analogous to the reflexivity of the Banach spaces is defined as

Definition 1.2.21: A Riesz space X is said to be *order perfect* if

the canonical mapping $J:X \longrightarrow (X^{so})^{so}$ defined by

$$J(x)(f) = f(x), \; \forall \; f \in X^{so} \text{ and } x \in X,$$

is onto ,where $(X^{so})^{so}$ is the sequential order dual of X^{so}.

Observe that the mapping J, in general, is a Riesz homomorphism from X into $(X^{so})^{so}$.

Coming back to o-linear mappings, we first quote a result from [229], which relates the order convergence of linear operators in $\mathcal{L}^b(X,Y)$ with the pointwise convergence as follows :

Theorem 1.2.22: With X and Y as in Theorem 1.2.20 , consider a net $\{T_\alpha : \alpha \in \Lambda\}$ in $\mathcal{L}^b(X,Y)$. Then

(i) $T_\alpha \xrightarrow{(o)} T$ in $\mathcal{L}^b(X,Y)$, implies that $T_\alpha(x) \xrightarrow{(o)} T(x)$ in Y, for each $x \in X$; and conversely

(ii) if $\{T_\alpha\}$ is an increasing or a decreasing net in $\mathcal{L}^b(X,Y)$ such that $T_\alpha(x) \xrightarrow{(o)} T(x)$ in Y, for each x in X, then $T \in \mathcal{L}^b(X,Y)$ and $T_\alpha \xrightarrow{(o)} T$ in $\mathcal{L}^b(X,Y)$.

The next proposition enunciated from [175] states the connection among order bounded , sequentially order continuous and order continuous linear operators.

Proposition 1.2.23:(i) Every sequentially order continuous linear operator from an Archimedean Riesz space into a Riesz space with boundedness property, is order bounded. In particular, every sequentially order continuous linear functional on an Archimedean Riesz space is order bounded.

(ii) Every sequentially order continuous linear operator from an order separable Riesz space into an Archimedean Riesz space is order continuous.

Characterizing Riesz homomorphisms, we have [6]

Proposition 1.2.24: A one to one positive, onto map T in $\mathcal{L}(X,Y)$

is a Riesz homomorphism if and only if T^{-1} is positive. A Riesz homomorphism T from X onto Y is normal if and only if the kernel N_T of T is a band in X.

Recalling the notation P_B for a projection band B from the note after the remark following Definition 1.2.12, it is possible to define a linear operator P_B from X into itself as follows :

Definition 1.2.25 : For $x \in X$ with $x = x^+ - x^-$, and a projection band B of X, define $P_B : X \longrightarrow X$ by

$$P_B(x) = P_B(x^+) - P_B(x^-).$$

Remark: Observe that from the definition of a projection band B, each $x \in X$ can be written uniquely as $x_1 + x_2$ where $x_1 \in B$ and $x_2 \in B^\perp$. As $B^\perp$ is also a band in X, in terms of the operators P_B and $P_{B^\perp}$, we have

$$x_1 = P_B(x) \text{ and } x_2 = P_{B^\perp}(x).$$

Simple facts concerning the operator $P \equiv P_B$ are summarized in [229].

Proposition 1.2.26: P is a normal Riesz homomorphism from X into itself such that

(i) $P(x) = x$ if and only if $x \in B$; $P(x) = \theta$ if and only if $x \perp B$; and

(ii) $\theta \leq P(x) \leq x$, $\forall\, x \in K$.

Defining

$$(1.2.27) \qquad p_A(x) = \sup_{f \in A} |\langle x, f \rangle|,$$

for a subset A of X' and $x \in X$, we recall the following concepts from [46],[47].

Definition 1.2.28 : A subset A of X' is said to satisfy the *condition* (i) A_1 if $A \subset X^{so}$ and $p_A(x_n) \longrightarrow 0$ as $n \longrightarrow \infty$, for every

sequence $\{x_n\} \subset X$ such that $x_n \xrightarrow{(o)} \theta$ in X ; (ii) A_2 if $A \subset X^c$ and $p_A(x_\alpha) \longrightarrow 0$ for every net $\{x_\alpha\}$ in X with $x_\alpha \xrightarrow{(o)} \theta$ in X ; and (iii) A_3 if $A \subset X^b$ and $p_A(x_n - x_m) \longrightarrow 0$ as $n,m \longrightarrow \infty$, for every majorized increasing sequence $\{x_n\}$ in X.

The sets satisfying conditions A_1, A_2 and A_3 are also known as *equi-σ-continuous* , *equicontinuous and equi-l^1-continuous* respectively ; cf.[27].

1.3 Topological Vector Spaces: In this section , we consider the particular cases of topological vector spaces (TVS) , namely, locally convex spaces (l.c.s.) , ordered topological vector spaces and locally solid Riesz spaces , which we denote by (X,τ), where the topology τ is generated by the family $\mathfrak{D}_\tau \equiv \mathfrak{D}$ of pseudonorms or seminorms.

Locally Convex Spaces: We refer to [86] for definitions, notations and results of this subsection . Let us denote by $\mathcal{U}_X$, the fundamental neighbourhood system consisting of balanced, convex and absorbing neighbourhoods at the origin θ of a locally convex space (X,τ) and by X^*, the topological dual of the space .

For a dual pair $\langle X,Y\rangle$ of vector spaces (we will consider in our discussion the real case only) , we denote by $\sigma(X,Y\rangle$, the weak topology on X , the locally convex topology generated by the family $\{p_y : y \in Y\}$ of seminorms , where $p_y(x) = |\langle x,y\rangle|$ for $x \in X$. If ζ is a collection of $\sigma\langle Y,X\rangle$-bounded subsets of Y, then we have the ζ-topologies, known as the *topologies of uniform convergence on members of* ζ and generated by the family $\{p_A : A \in \zeta\}$ of seminorms, where

$$p_A(x) = \sup_{y \in A} |\langle x,y\rangle|, \; x \in X.$$

In particular cases, when ζ is the collection of all balanced,

convex ,$\sigma(Y,X)$-compact subsets of Y (resp. balanced, convex and $\sigma(Y,X)$-bounded subsets of Y), we get the Mackey topology $\tau(X,Y)$ (resp. the strong topology $\beta(X,Y)$).

Recalling that a locally convex topology τ on X is *compatible* with the dual pair $\langle X,Y\rangle$ if $(X,\tau)^{*}= Y$, we have

Proposition 1.3.1: A locally convex topology τ on X is compatible with the dual pair $\langle X,Y\rangle$ if and only if τ is an ζ - topology where ζ is a collection of balanced, convex, $\sigma(Y,X)$-compact subsets of Y, which covers Y or equivalently $\sigma(X,Y) \subset \tau \subset \tau(X,Y)$.

A characterization of equicontinuous subsets of X^{*}, used in the sequel , is given in

Proposition 1.3.2: A subset M of X^{*} is equicontinuous if and only if $M \subset V^{\circ}$ for some $V \in \mathcal{U}_X$, where $V^{\circ}= \{ f \in X^{*} : |f(x)| \leq 1,$ for each x in V $\}$.

For a locally convex space (X,τ), the *symbol* $\lambda(X^{*},X)$ denotes an ζ -topology, where ζ is the collection of all precompact subsets of X. This topology is related with $\sigma(X^{*},X)$ in the sense of

Theorem 1.3.3 (Banach-Diéudonné):Let (X,τ) be a metrizable l.c.s. with dual X^{*}. Then $\lambda(X^{*},X)$ is the finest locally convex topology on X^{*} which induces on every equicontinuous subset of X^{*} , the same topology as $\sigma(X^{*},X)$.

If T is a linear operator from a vector space X to another vector space Y, then we write T^{*} for the adjoint of T, which is a map from Y' to X'. Concerning these maps T and T^{*}, we have

Proposition 1.3.4: If (X,τ) and (Y,S) are two locally convex Hausdorff spaces and $T:X \longrightarrow Y$ is τ-S continuous, then it is

$\sigma(X,X^*)-\sigma(Y,Y^*)$ continuous ; indeed, it is $\tau(X,X^*)$ - $\tau(Y,Y^*)$ continuous. Further, T^* is $\sigma(Y^*,Y)-\sigma(X^*,X)$ continuous .

Ordered Topological Vector Spaces: An ordered vector space X equipped with a linear topology τ is known as an *ordered topological vector space (OTVS)* and if τ is also a Hausdorff locally convex topology , it is called an *ordered locally convex space (o.l.c.s.)*. The concepts of ordering and topological structures of X are usually related via cones of an OVS . Accordingly, we have different types of cones in an OTVS , defined in

Definition 1.3.5 : A cone K in an OTVS (X,τ) is said to be (i) *regular, w-regular* (resp. *fully regular* , *fully w-regular*) if every order bounded (resp. τ -bounded) increasing net , sequence in K converges to an element of K ; (ii) *minihedral* if for x,y in K , $x \vee y$ exists in K; (iii) *normal* if there exists a neighbourhood basis of θ consisting of full sets , where $A \subset X$ is said to be *full* if $A = [A]$, where $[A] = (A+K) \cap (A-K)$; (iv)an ζ *-cone* (resp. a *strict* ζ *-cone)* for a saturated class ζ of τ-bounded subsets of X with $X = \cup\{S : S \in \zeta\}$, if the class $\bar{\zeta}_k = \{\ \overline{(S\cap K)-(S\cap K)} : S \in \zeta\ \}$ (resp. $\zeta_k = \{(S\cap K)-(S\cap K) : S \in \zeta\ \}$) is a fundamental system for ζ ; (v) a *b-cone* (resp. *a strict b-cone*) if it is an ζ-cone (resp. a strict ζ -cone) where ζ is the class of all bounded subsets of (X,τ).

Normal cones in an OTVS or o.l.c.s. , play a significant role in various applications of the theory of OTVS; indeed, the cones of locally solid Riesz spaces are always normal. A Characterization of such cones in terms of seminorms is given in

Proposition 1.3.6: The cone of an o.l.c.s. (X,τ) is normal if and only if there exists a family $\{p_\alpha : \alpha \in \Lambda\}$ of seminorms generating the topology τ such that $\theta \le x \le y$ implies that $p_\alpha(x) \le p_\alpha(y)$ for all $\alpha \in \Lambda$, (such seminorms are known as *monotone seminorms*).

It is well known [143] that in an ordered *Fréchét space* (i.e., complete metrizable o.l.c.s.) (X,τ) , a cone is normal if and only if it is weakly normal and in an ordered *F-space* (i.e., complete metrizable OTVS) , K is normal if and only if $[\theta,x]$ is bounded in (X,τ) for each x in K.

Locally Solid Riesz Spaces: Locally solid Riesz spaces are nothing but particular type of OTVS, where the underlying space is a Riesz space and the topology is restricted in the following sense :

Definition 1.3.7 :A *locally solid (l.s.) Riesz space* (resp.a *locally convex solid (l.c.s.) Riesz space*) is an OTVS (resp. o.l.c.s.) for which the neighbourhood system $\mathcal{U}_X$ consists of solid sets and the topology τ of l.s. Riesz space (resp. a l.c.s. Riesz space) is called a locally solid (resp. a locally convex solid) topology. In case τ is a normed topology , the space (X,τ) is known as a *normed lattice* and it is referred to as a *Banach lattice* if it is a complete normed lattice.

A characterization of a l.s. Riesz space is given in [47]

Proposition 1.3.8: A locally convex topology τ on X is locally solid if and only if X^* is an ideal in X^b and solid hull of an equicontinuous subset of X^* is equicontinuous, or equivalently, there exists a family $\mathfrak{D}$ of *Riesz seminorms* (i.e.,$p(x) \le p(y)$ for $|x| \le |y|$) generating the topology τ.

Some properties of a Hausdorff locally solid Riesz space (X,τ), are listed in

Proposition 1.3.9: For the space (X,τ) as above, we have

(i) X is Archimedean.

(ii) The cone K is τ-closed.

(iii) If $x_\alpha \longrightarrow x$ in τ and $x_\alpha \uparrow$ (resp. $x_\alpha \downarrow$) in X, then $x_\alpha \uparrow x$ (resp. $x_\alpha \downarrow x$) in X.

Various kinds of locally solid Riesz spaces used in the sequel are defined in

Definition 1.3.10: A l.s. Riesz space (X,τ) satisfies (i) *σ-Lebesgue* property (or A_1) if $x_n \downarrow \theta$ in X implies $x_n \xrightarrow{(\tau)} \theta$; (ii) *Lebesgue* property (or A_2) if $x_\alpha \downarrow \theta$ in X implies that $x_\alpha \xrightarrow{\tau} \theta$; (iii) *pre-Lebesgue* property (or A_3) if $\theta \le x_n \uparrow \le x$ in X yields that $\{x_n\}$ is a τ - Cauchy sequence in X ; and (iv) *Fatou property* (resp. *σ-Fatou property*) if there exists a neigbhourhood basis of order closed *(resp. σ-order closed)* sets.

The topology τ of the space (X,τ) satisfying the property in (i), (ii), (iii) and (iv) is respectively referred to as the *σ-Lebesgue*, *Lebesgue*, *pre-Lebesgue* and *Fatou* *(resp. σ-Fatou)* topology.

Depending on the concepts of the above definition , we have several types of Riesz seminorms defined as

Definition 1.3.11 : A Riesz seminorm p on X is said to be (i) *σ-Lebesgue seminorm* if $p(x_n) \longrightarrow 0$, for any sequence $x_n \downarrow \theta$ in X ; (ii) *Lebesgue seminorm* if $p(x_\alpha) \longrightarrow 0$ for any net $x_\alpha \downarrow \theta$ in X ; (iii) *pre-Lebesgue seminorm* if for any disjoint order bounded sequence $\{x_n\}$ in X , $p(x_n\} \longrightarrow 0$; and (iv) *Fatou seminorm* (resp. *σ-Fatou seminorm*)if $\theta \le x_\alpha \uparrow x$ (resp.$\theta \le x_n \uparrow x$) in X yields that

$p(x_\alpha) \uparrow p(x)$ (resp. $p(x_n) \uparrow p(x)$).

Characterizing Fatou topology and pre-Lebesgue topology, we have [6]

Proposition 1.3.12: Let (X,τ) be a l.c.s. Riesz space. Then

(i) τ is a Fatou topology if and only if there exists a family $\{p_\alpha\}$ of Fatou seminorms generating the topology τ ; and

(ii) (X,τ) satisfies pre-Lebesgue property if and only if every disjoint order bounded sequence in X is τ- convergent to zero.

Concerning the locally solid character of Mackey topologies on X, we have [47]

Proposition 1.3.13 : For a Riesz space X, (i) the Mackey topology $\tau(X,X^b)$ is always locally solid ; and (ii) $\tau(X,X^{so})$ (resp. $\tau(X,X^c)$) is locally solid if and only if it satisfies A_1(resp. A_2).

The *order topology* $o(X,Y)$ or $|\sigma|(X,Y)$ (termed as *absolute weak topology*) defined corresponding to a dual pair $\langle X,Y\rangle$ of Riesz spaces ,where Y is an ideal in X^b ,is the topology of uniform convergence on all order intervals of Y, or equivalently, it is defined by the Riesz seminorms $\{p_y : y \in Y\}$, where $p_y(x) = \langle |x|,|y|\rangle$; and the *topology* τ_o is the finest locally convex topology on X for which every order bounded set is topologically bounded .

Concerning these topologies , we quote from [6], [175]

Proposition 1.3.14 : (i) The dual of X equipped with $o(X,Y)$ is Y.

(ii) In an Hausdorff l.c.s. Riesz space (X,τ), $\sigma(X,X^*) = |\sigma|(X,X^*)$ if and only if every order interval of X^* is contained in a finite dimensional vector space .

(iii) The dual of (X,τ_o) is X^b ; and if (X,τ) is a complete metrizable locally solid Riesz space , then τ coincides with τ_o on X.

Restricting the order bounded sets in a l.s. Riesz space (X,τ), Duhoux [48] introduced

Definition 1.3.15 : A set A in X is said to be *quasi-order precompact* (resp. *order- precompact*) if for each neighbourhood U of θ , there exists a positive element x in X (resp. in the ideal generated by A in X) such that $A \subset [-x,x] + U$.

Clearly, every order-precompact set is quasi-order precompact. The converse is not necessarily true, e.g. the set $\{e_n : n \geq 1\}$ [cf. (1.4.1) for the definition] is quasi-order precompact, but it is not order-precompact in ℓ^∞ . However, we have [48]

Proposition 1.3.16 : The topology τ of a l.c.s. Riesz space (X,τ) is pre-Lebesgue if and only if every quasi -order precompact set in (X,τ) is order precompact.

According to [48], the o-precompactness of operators defined from a Riesz space X to a l.c.s. Riesz space (Y,τ), means

Definition 1.3.17 : $T \in \mathcal{L}^b(X,Y)$, where X is a Riesz space and (Y,τ) is an order complete l.c.s. Riesz space , is said to be *o-precompact* if it maps order bounded subsets of X to precompact subsets of Y.

The space of all o-precompact operators from X to Y is denoted by $\mathcal{L}_{op}(X,Y)$ and for this space , we have [7], [48].

Proposition 1.3.18 : $\mathcal{L}_{op}(X,Y)$ forms a band in $\mathcal{L}^b(X,Y)$ where X and Y are as in Definition 1.3.17 with (Y,τ) having Lebesgue property.

Lastly in this section we consider concepts related to Banach lattices from [6]

Definition 1.3.19 : (i) A Riesz seminorm p on a Riesz space X is known as *m -additive*, $m \geq 1$, if $p(x+y)^m = p(x)^m + p(y)^m$, for every $x, y \in K$ with $x \wedge y = \theta$; and (ii) a Banach lattice X with Riesz norm p, is called an *abstract L-space* if p is 1-additive.

Theorem 1.3.20 : (i) Every abstract L-space is order complete and satisfies the Lebesgue property.

(ii) Every Banach lattice X satisfies $X^* = X^b$.

The well known normed lattices with m - additive norms are the classical Banach lattices $L_m[0,1]$; cf.[6], p.71.

1.4 Sequence Spaces : By a sequence space we mean a vector space whose elements are sequences from a vector space X which may also be $\mathbb{R}$ or $\mathbb{C}$. In case X is different from $\mathbb{R}$ or $\mathbb{C}$, a sequence space is referred to a *vector valued sequence space* (VVSS) , otherwise it is known as a *scalar valued sequence space* (SVSS). As our study is confined only to real vector spaces , we shall be considering $\mathbb{R}$ only for SVSS.

Scalar Valued Sequence Spaces: We follow [96] for various notations and results for the theory of SVSS. We denote by w the space of all real sequences and ϕ, the subspace of w, spanned by $\{e^n : n \geq 1\}$, where

$$(1.4.1) \qquad e^n = \{0,0,\ldots,\underset{\text{n-th place}}{1},0,0,\ldots\}, \; n \in \mathbb{N}.$$

Elements of w are denoted by $\bar{\alpha} = \{\alpha_i\}$, $\bar{\beta} = \{\beta_i\}$ etc. and n^{th} *section* $\alpha^{(n)}$ of an element $\bar{\alpha} = \{\alpha_i\}$ is defined as $\alpha^{(n)} = \sum_{i=1}^{n} \alpha_i e^i$.

The element $\{1,1,1,\ldots.\}$ *in* w *is denoted by* e.

An *SVSS* means a subspace λ of w , containing ϕ. The *Köthe* and *β duals* of λ are respectively defined as

$$\lambda^{x} = \{ \{\beta_n\} \in w : \sum_{n=1}^{\infty} |\alpha_n\beta_n| < \infty , \forall \{\alpha_n\} \in \lambda\} ,$$

$$\lambda^{\beta} = \{\{\beta_n\} \in w: \sum_{n=1}^{\infty} \alpha_n\beta_n < \infty, \forall \{\alpha_n\} \in \lambda\}.$$

The space λ is in duality with λ^{x} as well as λ^{β} corresponding to the bilinear form $\langle\ ,\ \rangle$, where, for $\bar{\alpha} = \{\alpha_n\} \in \lambda$ and $\bar{\beta} = \{\beta_n\}$ in λ^{x} or λ^{β},

$$\langle\bar{\alpha}, \bar{\beta}\rangle = \sum_{n=1}^{\infty} \alpha_n\beta_n .$$

The *weak topology* $\sigma(\lambda,\lambda^{x})$ and the *normal topology* $\eta(\lambda,\lambda^{x})$ on λ are respectively generated by the families $\{ q_{\bar{\beta}} : \bar{\beta} \in \lambda^{x}\}$ and $\{ p_{\bar{\beta}} : \bar{\beta} \in \lambda^{x}\}$ of seminorms, where

(1.4.2) $$q_{\bar{\beta}}(\bar{\alpha}) = |\sum_{n=1}^{\infty} \alpha_n\beta_n| \qquad \text{and} \qquad p_{\bar{\beta}}(\bar{\alpha}) = \sum_{n=1}^{\infty} |\alpha_n\beta_n| .$$

A characterization of $\sigma(w ,\phi)$- bounded subset of w, used in our later work is

Proposition 1.4.3 : A subset B of w is $\sigma(w,\phi)$ - bounded if and only if there exists a sequence $\{r_i\} \subset \mathbb{R}_+$ such that $|x_i| \le r_i$, for each $i \ge 1$, and for all $\bar{x} = \{x_i\} \in B$.

A sequence space λ is *monotone* (resp. *normal*) if $\{\alpha_n x_n\} \in \lambda$ whenever $\{x_n\} \in \lambda$, and $\alpha_n = 0$ or 1 (resp. $|\alpha_n| \le 1$), $\forall\ n \ge 1$. A sequence space λ equipped with a locally convex topology $\mathcal{T}$, is an *AK- space* if $\mathcal{T}$ is finer than $\sigma(\lambda, \phi)$ and for each $\bar{\alpha} \in \lambda$, $\alpha^{(n)} \longrightarrow \bar{\alpha}$ in $\mathcal{T}$ and an AK-space $(\lambda,\mathcal{T})$ where $\mathcal{T}$ is a complete metrizable locally convex topology, is known as a *Fréchét AK-space* .

Concerning the dual of a topological sequence space $(\lambda,\mathcal{T})$, we have

Proposition 1.4.4 : The topological dual λ^{*} of a sequence space λ equipped with a locally convex topology $\mathcal{T}$, is λ^{β}, provided $(\lambda,\mathcal{T})$ is a barrelled AK-space .

For an infinite matrix $A \equiv [a_{ij}]$, let us recall from [96] the following :

Definition 1.4.5 : An infinite matrix $A \equiv [a_{ij}]$ is said to be a *matrix transformation* from a sequence space λ into another sequence space μ provided , for each $\bar{\alpha}=\{\alpha_j\} \in \lambda$, the series $\sum_{j=1}^{\infty} a_{ij}\alpha_j$ converges absolutely for each $i\geq 1$; and the sequence $\bar{\beta} = \{\beta_i\}$ where $\beta_i = \sum_{j=1}^{\infty} a_{ij}\alpha_j, i\geq 1$, is an element of μ .

A characterization of the matrix representation of a linear map in terms of continuity, is contained in [96]

Proposition 1.4.6 : Let λ and μ be two sequence spaces such that λ is monotone . Then a linear transformation A from λ to μ is a matrix transformation if and only if it is $\sigma(\lambda,\lambda^{x})$ – $\sigma(\mu,\mu^{x})$ continuous .

As the real line is a Riesz space , the SVSS automatically becomes an OVS with respect to the co-ordinate wise ordering induced from the ordering of $\mathbb{R}$,i.e., $\bar{\alpha} \leq \bar{\beta}$ if and only if $\alpha_i \leq \beta_i$, $\forall$ $i \in \mathbb{N}$,a fact being used throughout our work for SVSS. Concerning the order structure of λ , we have [176].

Proposition 1.4.7 : (i) The sequence spaces ϕ and w are Riesz spaces and every sequence space λ is an order separable ordered vector space .

(ii) Let a sequence space λ be an ideal in w. Then a linear functional $\bar{f}$ on λ is sequentially order continuous if and only if there is a unique $\bar{u} =\{u_n\} \in \lambda^{x}$ such that

$$\bar{f}(\bar{\alpha}) = \langle\bar{\alpha} ,\bar{u}\rangle = \sum_{n=1}^{\infty} \alpha_n u_n ,$$

for all $\bar{\alpha} =\{\alpha_n\}$ in λ .

Note 1.4.8: In view of Propositions 1.2.23(ii) and 1.4.7(i), it

follows from Proposition 1.4.7(ii) that

$$\lambda^{x} \equiv \lambda^{so} \equiv \lambda^{c}.$$

<u>Vector Valued Sequence Spaces</u> : For the literature of this subsection , we refer to [37], [72], [170], [190].

Let $\langle X,Y\rangle$ be a dual pair of real vector spaces . By $\Omega(X)$ and $\phi(X)$, we mean the vector space of all sequences of X and the subspace of $\Omega(X)$, consisting of all finitely non-zero sequences respectively. For $x \in X$ and $\bar{x}=\{x_i\} \in \Omega(X)$, we write

$$\delta_i^{\,x} = \{\theta,\theta,\ldots,\underset{i\text{-th place}}{x},\theta,\theta,\ldots\} \ , \quad i\in \mathbb{N} \ ;$$

and

$$\bar{x}^{(n)} = \sum_{i=1}^{n} \delta_i^{\,x_i} \qquad , \ n \in \mathbb{N} \ .$$

A VVSS or a *generalized sequence space* $\Lambda(X)$ is a vector space of sequences from X containing $\phi(X)$. The *generalized Köthe dual* $\Lambda^{x}(Y)$ and the *generalized β-dual* $\Lambda^{\beta}(Y)$ of $\Lambda(X)$ are the spaces defined as

$$\Lambda^{x}(Y) = \{ \{y_i\}\in w : \sum_{i=1}^{\infty} |\langle x_i,y_i\rangle| \text{ converges}, \ \forall \ \bar{x} = \{x_i\} \in \Lambda(X)\};$$

$$\Lambda^{\beta}(Y) = \{ \{y_i\}\in w : \sum_{i=1}^{\infty} \langle x_i,y_i\rangle \text{ converges}, \ \forall \ \bar{x} = \{x_i\} \in \Lambda(X)\}.$$

The generalized Köthe dual of $\Lambda^{x}(Y)$ is denoted by $\Lambda^{xx}(X)$ and so on.

A VVSS $\Lambda(X)$ is said to be *perfect* if $\Lambda^{xx}(X)=\Lambda(X)$. It is *normal* if $\ell^{\infty}\Lambda(X) \subset \Lambda(X)$; and *monotone* if $m_o\Lambda(X) \subset \Lambda(X)$, where m_o is the vector space spanned by all sequences having co-ordinates equal to zero or one . *Normal hull* of a set $B \subset \Lambda(X)$ is the collection of all $\bar{y}=\{y_i\}\in \Lambda(X)$ such that $y_i=\alpha_i x_i$, for some $\bar{x}=\{x_i\} \in \Lambda(X)$ and some sequence $\{\alpha_i\}$ of scalars with $|\alpha_i| \le 1$, $\forall$ $i\ge 1$. A VVSS $\Lambda(X)$ equipped with a locally convex topology $\mathfrak{F}$, is said to be *simple* if each $\mathfrak{F}$ bounded set is contained in the

normal hull of a singleton set .

Clearly every perfect VVSS is normal and every normal VVSS is monotone . Also, we have [170]

Proposition 1.4.9 : For a monotone VVSS $\Lambda(X)$, $\Lambda^{x}(Y) = \Lambda^{\beta}(Y)$.

The space $\Lambda(X)$ is in duality with $\Lambda^{x}(Y)$ and $\Lambda^{\beta}(Y)$ with respect to the bilinear form B defined by

$$B(\bar{x},\bar{y}) = \sum_{i=1}^{\infty} \langle x_i , y_i \rangle, \ \{x_i\} \in \Lambda(X), \ \{y_i\} \in \Lambda^{x}(Y) \text{ or } \Lambda^{\beta}(Y);$$

and the various polar topologies , namely, weak topologies , strong topologies and Mackey topologies defined on either of the spaces $\Lambda(X)$ or $\Lambda^{x}(Y)$ ($\Lambda^{\beta}(Y)$) are Hausdorff locally convex topologies.

Concerning the perfectness of the spaces $\phi(X)$ and $\Omega(X)$, we have [190]

Proposition 1.4.10 : For a dual pair $\langle X,Y \rangle$ of vector spaces X and Y,

(i) $[\phi(Y)]^{x} = \Omega(X)$, $[\Omega(X)]^{x} = \phi(Y)$; and

(ii) $[\phi(X)]^{x} = \Omega(Y)$, $[\Omega(Y)]^{x} = \phi(X)$.

Hence $\phi(X)$ and $\Omega(X)$ are perfect spaces .

Another type of VVSS which is being discussed in our work, is formed with the help of a normal SVSS λ and a locally convex space (X,τ) and is defined as

(1.4.11) $\lambda(X) = \{ \bar{x} = \{x_i\} : x_i \in X, i \geq 1 \text{ and } \{p(x_i)\} \in \lambda, \text{ for all } p \in \mathfrak{D} \}$.

To topologize $\lambda(X)$, we assume that (X,τ) is a Hausdorff l.c.s., and λ is equipped with a normal ζ -topology T_λ [170] compatible with the dual pair $\langle \lambda, \lambda^{x} \rangle$ and generated by the seminorms $\{p_S : S \in \zeta\}$, where $p_S(\bar{\alpha}) = \sup_{\{\beta_i\} \in S} \{ \sum_{i=1}^{\infty} |\alpha_i||\beta_i| \}$, and ζ is a family of normal hulls of balanced , convex , $\sigma(\lambda^{x},\lambda)$-

bounded sets of λ^{X}, covering λ^{X}. For $S \in \zeta$, $p \in \mathcal{D}$ and $\bar{x}=\{x_i\}$ in $\lambda(X)$, we define

(1.4.12) $\quad P_S(\bar{x}) = (p_S \circ p)(\bar{x}) = p_S(\{p(x_i)\})$.

Then $\{ p_S \circ p : S \in \zeta, p \in \mathcal{D}\}$ is a family of seminorms generating a Hausdorff locally convex topology $T_{\lambda(X)}$ on $\lambda(X)$. If λ and X are normed spaces, then $T_{\lambda(X)}$ is a normed topology for the norm $\|\ \|_{\lambda(X)}$ defined by

$$\|\bar{x}\|_{\lambda(X)} = \|\{\|x_i\|_X\}\|_{\lambda} .$$

CHAPTER 2

HISTORY AND MCTIVATION

2.1 An Overview : This chapter which is divided into four sections dealing respectively with ordered vector spaces and Riesz spaces , locally solid Riesz spaces , vector valued sequence spaces and linear operators on Riesz spaces , presents a scenario of developments of various branches of Riesz spaces; though the treatment is not exhaustive by all means.

2.2 Ordered Vector Spaces and Riesz Spaces: These spaces are real vector spaces equipped with certain type of partial ordering that is compatible with the algebraic structure of the space, and have their origin in an address of F. Riesz at International Congress of Mathematicians held at Bologona in the year 1928 [193], where he considered the partial ordering on the class of linear functionals defined on the class of functions. An extended version of his note appeared in the year 1940 in his later work [194] which is a translation of his 1937 Hungarian paper. Indeed, in [194] he replaced the class of functions by a fundamental set Ω , by which he meant an arbitrary abelian group relative to the operation of addition , with the exception of negative elements such that the addition also satisfies the cancellation law, decomposition property (cf.[175],p.7) and sum of two elements of Ω is zero only if each of them is zero; and proved the order complete lattice structure of the class containing the difference of two additive positive functionals on Ω.

Almost simultaneously , the Dutch mathematician H. Freudenthal, influenced by Riesz's address at Bologona, got his

work [56] published in the year 1936 on a partially ordered module, a concept equivalent to that of a Riesz space. However, a systematic treatment of the theory of partially ordered vector spaces dealing with the algebraic and convergence properties of Riesz spaces with its applications to linear operator theory was broughtforward by L.V.Kantorovich during the year 1935 to 1937 in the papers [99, 100, 101, 102, 103, 104]. His investigations on this theory attracted the attention of Leningrad school of mathematicians, namely, A.G. Pinsker, A.I. Yudin, G.P.Akilov, A.N.Baluyev and B.Z.Vulikh who contributed in the development of the theory of partially ordered vector spaces in various ways, e.g. finding conditions for the extension of linear operators from subspaces to the whole space, developing the structural properties of spaces or finding applications in the field of differential equations.

Almost at the same time , the general theory of lattices of which Riesz spaces are particular cases, was developed extensively by G.Birkhoff who presented the same in a systematic form in his monograph [20]. This work of Birkhoff facilitated L.Fuchs to publish a book [57] on partially ordered algebraic systems dealing mainly with the partially ordered groups, semigroups, rings and fields, where the positivity of an element was defined with the help of unit element of the group. A few years later in 1966, Fuchs got his lectures, delivered at Queen's University and based on his work appeared in [58, 59, 60], published in [61]. These notes were devoted to the study of Riesz vector spaces and Riesz algebras which are respectively ordered vector spaces and ordered algebras with Riesz

interpolation property, a notion equivalent to the decomposition property in Riesz spaces and the former being more general than Riesz spaces.

Because of the war conditions, Japanese mathematicians were unaware of the researches going on in the Soviet Union in the field of ordered vector spaces, and some of them, namely, Maeda [139], Maeda and Ogasawara [140] and Nakano [153, 154, 155], made important contributions independently in this theory. Whereas, Ogasawara's book [168] on vector lattices in Japanese appeared in the year 1948 , Nakano got his work published in the form of books [156, 157] in 1950. In the same year, the monograph on functional analysis in partially ordered spaces by L.V.Kantorovich, B.Z.Vulikh and A.G.Pinsker [105] appeared in Russian, in which the theory of K-spaces (i.e.,order complete lattices) was discussed extensively. Indeed, they considered order-convergence, (*)-convergence, decomposition of a K-space as direct sum of pairwise disjoint bands and various other results relating extension, representation of operators on K-spaces; besides incorporating the work of Freudenthal [56] on integral representation of elements of K-spaces and that of S.Kakutani [94, 95] on L- and M- spaces, and several other results dealing with the moment problems etc. We shall now sketch in brief the historical development of these various concepts in the following few paragraphs.

The concept of order convergence introduced around the year 1935 by Kantorovich [99] and Birkhoff [19], plays an important role in the study of Riesz spaces, especially in defining various other concepts of convergence, order continuous

linear operators etc. Indeed the definition of order (*)-convergence and relative uniform convergence (ru-convergence) as given in Definition 1.2.8 make use of the notion of order convergence; though these concepts for arbitrary convergence were introduced respectively by P.S.Aleksandrov, P.S.Uryson [9] and E.H.Moore [146] in the year 1923 and 1912. All these concepts of convergence coincide in the case of real line $\mathbb{R}$; though in general it may not be true; for example, the sequence $\{\bar{\alpha}^n\}$ in ℓ^∞ where $\alpha_i^n = 0$, for $i \le n$ and $=1$ for $i > n$, order converges to θ, but it does not relatively uniformly converge to θ; cf.[175],p,48. Indeed, a property of convergence in a metrizable TVS X , namely, for a null sequence $\{x_n\}$ in X, there exists a sequence of numbers $\lambda_n \uparrow \infty$ such that $\{\lambda_n x_n\}$ is a null sequence, cf. [92],p 42, plays a crucial role for the equivalence of these concepts in $\mathbb{R}$. This property in Riesz spaces is known as stability of convergence and it was shown by Zaanen [247] that

" In an Archimedean Riesz space, order convergence coincides with ru-convergence for sequences if and only if order convergence is stable."

He further proved

" In an Archimedean Riesz space X with a strong unit (an element in a Riesz space X is a strong unit if X is the ideal generated by it), which is either σ-order complete or possesses projection property, order convergence is stable if and only if it is of finite dimension ."

On the other hand, these convergence concepts when considered in specific Riesz spaces, have definite meanings. In fact, for scalar valued sequence spaces, order convergence

coincides with the co-ordinatewise convergence in the sense that a net $\{\bar{x}_\alpha : \alpha\in\Lambda\}$ in a sequence space λ order converges to $\bar{x}^{(o)}$ in λ if and only if it is order bounded and converges co-ordinatewise to $\bar{x}^{(o)}$, a result proved by Peressini and Sherbert [176] around the year 1966. For the function space S consisting of all real valued, measurable, almost everywhere finite functions in the closed interval [a,b], we have the Riesz theorem stating that the (*)-convergence in S is equivalent to the convergence in measure. In the space C(X) (the space of real continuous functions on a compact set X) and ℓ^∞, ru-convergence coincides with the uniform convergence, i.e., with the natural topological convergence.

Coming to the discussion of subspaces of a Riesz space X, we begin with the notion of a band which is defined as an order-closed ideal in X, cf. Definition 1.2.9. Indeed, this concept was introduced by Riesz [193, 194] under the nomenclature of 'complete family ' according to whom it is the class of real valued mappings closed with respect to finite sum, minorants and supremum of every majorized subset. In [105] and [156, 157], bands referred to as components and normal manifolds respectively, were introduced in an arbitrary Riesz space, possessing the property of an ideal and closed with respect to the supremum of any majorized subset. For the particular order complete Riesz space X of all additive set functions of bounded variation defined on a ring formed by subsets of a fixed set, K.Yosida and E.Hewitt [244] and H.Bauer [15] proved that the set of all countably additive set functions in X forms a band . Replacing the space X by the dual B^* of a Banach space B consisting of functions defined on an abstract set S and

following the line of action of [244], H.Gordon and E.R.Lorch [69] considered the projection operator from B^* to the class of integrals; where by an integral they meant a linear functional F on B ,written as the difference of two positive linear functionals F_1 and F_2 such that for any sequence $\{f_n\} \subset B$, $F_i(f_n) \longrightarrow F_i(f)$, $i = 1,2$, whenever $f_n \uparrow f$ pointwise in B; indeed, this class of integrals was shown to be a band in B^*. In 1960, Gordon [67] independently generalized his work with Lorch [69] by considering the order dual X^+ of an arbitrary Riesz space X and proved the existence of a band B in X^+ through the projection operator from X^+ to B , where B is the class of atomic functionals on X, where a functional on X is atomic if it is sum of functionals each of which can not be expressed in a non trivial way as the sum of smaller functionals. The investigations dealing with the projection property and projection bands in a Riesz space continued and we may cite a few names, say, C.B.Huijsmas [87], J. Jakubik [88], A.I.Veksler [227] and E.P.De Jonge [38, 39], who respectively obtained certain equivalent conditions for an ideal to become a projection band, related principal projection bands with the property of possessing sufficiently many projections in a Riesz space, studied weak and strong Freudenthal properties of positive elements in an Archimedean Riesz space X [$u \in K$ in X has weak (resp. strong) Freudenthal property if $x = \sup \{z: \theta \le z \le x$, $z \in \text{span}\,\{y \in X: y \wedge (u-y) = \theta\}$, $\forall\, x \in B_u$ (resp. $\forall\, x \in I_{\{u\}}$, and $\varepsilon > 0$, there exist n_ε in $\mathbb{N}$ and an increasing sequence $\{x_n\}$ in span $\{ y \in X: y \wedge (u-y) = \theta\}$ such that $|x_n - x| \le \varepsilon\, u$, $\forall\, n \ge n_\varepsilon$)] and found characterization for an Archimedean Riesz space to have projection property.

A more general concept than that of the band is the notion of an ideal which appeared first in [94] and [105] in connection with the embedding of an abstract (L)-space (cf. Definition 1.3.19) and any order complete Riesz space respectively, as a subspace of C(X), possessing the property of an ideal, where X is an extremally disconnected compact Hausdorff space. In the process of characterizing the space of all real continuous functions on a compact Hausdorff space, as a partially ordered group, Ky Fan [122] studied these ideals as 'convex subgroups' H of a partially ordered group G, defined as $[h_1,h_2] \subset H$, for $h_1,h_2 \in H$; and proved under certain restrictions that $\mathbb{R}$ is the only non trivial OVS with no proper ideal. This result was also proved independently by R.V.Kadison [93] who considered an ideal as a linear subspace I of an OVS such that $x \in I$, whenever $-y \le x \le y$, for some y in I. Following the terminology of Kadison, F.F.Bonsall [22] improved their result by removing all the restrictions; and observed the relationship of maximal ideal with the normalized positive linear functional in the form

" For an OVS X with an order unit e, corresponding to any proper ideal I in X, there exists a maximal ideal M with $I \subset M$ and a normalized positive linear functional ϕ_M with M as its kernel.",

a result proved implicitly in [93]. Bonsall continued his investigations on ideals in an OVS in [24] and [25] and also found applications of his results to the solution of Stieltjes and Hamburger moment problems, cf. [25], p.631 and [235],p.125. Indeed, Bonsall's contributions on maximal, perfect (an ideal I in an OVS X with order unit e is perfect if for each x in I and

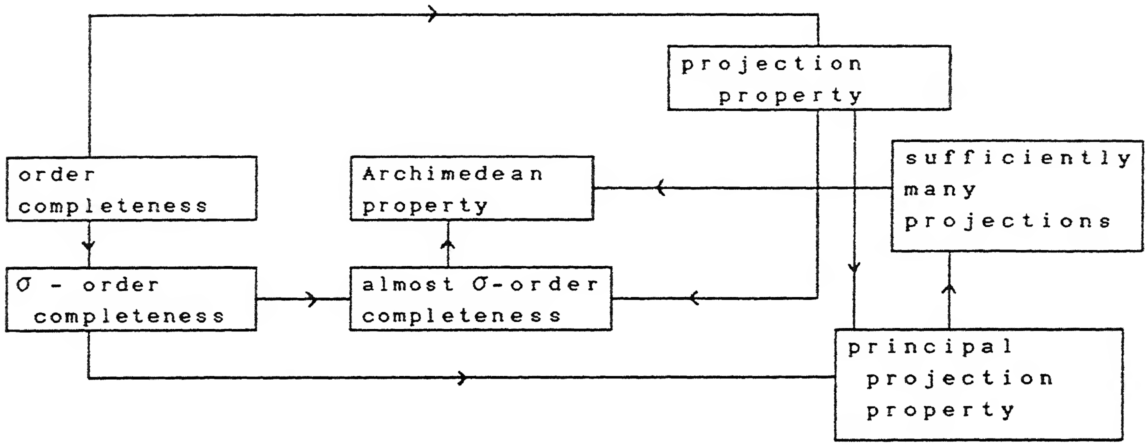
$\varepsilon>0$, there exists w_ε such that $-(\varepsilon e + w_\varepsilon) \le x \le (\varepsilon e + w_\varepsilon))$ and extreme maximal ideals (ideals for which ϕ_I is the extreme point of the convex set of all normalized positive linear functionals) were further further carried out by J.Kist [112] and Veksler [224] involving indecomposable maximal ideals (i.e., maximal ideal I which is the kernel of an indecomposable positive linear functional f, i.e., $\theta \le g \le f \Rightarrow g = \lambda f$, for some λ in $\mathbb{R}^+$) and maximal l-ideals. All the concepts of extreme ideals, perfect ideals, and indecomposable ideals coincide for a maximal lattice ideal (i.e. an ideal in our terminology, cf. Definition 1.2.9) in a Riesz space. On the other hand, G.Maltese [141] and K.E.Aubert [14] paid attention to find the correspondence between convex regular maximal ideals and positive multiplicative forms in a real partially ordered Banach algebra. The concept of an ideal in a Riesz space plays a very significant role in our study of sequence spaces to be discussed in the subsequent chapters

Besides ideal theory of Riesz spaces, another concept in Riesz spaces used in the sequel is the notion of order completeness. It was Kantorovich who first studied this property in Riesz spaces and such a Riesz space was later referred by Pinsker as a 'K-space', cf. [105], [183]. Following the Dedekind method of cuts for proving the completeness of the real number system $\mathbb{R}$, H.M.Mac Neill [136] in the year 1937, proved that any partially ordered set can be embedded in an order complete lattice with the preservation of suprema and infima and hence 'K-spaces' are also known as 'Dedekind complete' or 'order complete' Riesz spaces in the literature. Almost three and four years later, A.H.Clifford [30] and A.I.Yudin [245] published

papers dealing with the embedding of partially ordered groups and Archimedean Riesz spaces in order complete groups and order complete Riesz spaces respectively, the latter being known as the order or Dedekind completion of the given one. It has been pointed out in [189] that the Dedekind completion of a Riesz space as studied by Mac Neill [136] is not necessarily a Riesz space (in fact, it is a Riesz space if and only if the original space is Archimedean) and in the process of removing this shortcoming, following C.J.Everett [52], who introduced the concept of order Cauchy (i.e., o-regular according to his terminology) sequences in lattice modules, which was picked up by G.F.Papangelou [169] to study order Cauchy completion of commutative lattice groups, Quinn [189] first defined an intermediate Riesz space H with respect to a Riesz space X, which is a Riesz space for which there exists a Riesz isomorphism ϕ known as intermediate embedding, from X into H such that for each $y \in H$, $y = \sup\{ \phi(x): x \in X, \phi(x) \le y \} = \inf \{ \phi(x), x \in X, \phi(x) \ge y \}$. He then discussed various types of completions, namely σ-order completion, of an Archimedean Riesz space, order Cauchy completion,and relatively uniform completion of a Riesz space which are defined respectively as the smallest σ-order complete, order Cauchy complete and uniform complete intermediate Riesz space containing $\phi(X)$. The class of almost σ-order complete Riesz spaces (the spaces for which σ-order completion and order Cauchy completion coincide according to Quinn) was also introduced by C.D.Aliprantis and E.Langford [5] independently in 1974. The order Cauchy completion of the space C(X) of all continuous functions on a Tychnoff space X was investigated by F.K.Dashiell,

A.W.Hager and M.Henriksen in the year 1980 [35]. We have the following implication diagram dealing with these various types of completeness and projection properties proved by W.A.J.Luxemburg and A.C.Zaanen [135] and modified by Aliprantis and Langford [5] by adding almost σ-order completeness in between.

Diagram 2.1.1



On the other hand, using disjoint elements, S.Bernau [16] introduced the concepts of lateral completeness (i.e., every set containing disjoint elements has a supremum) and orthocompleteness (i.e., lateral completeness and $G = \{x\}^{\perp} \oplus \{x\}^{\perp\perp}$, for each element x in G) for a lattice group G, which were later studied by Veksler and V.A.Geiler [228] and Veksler [226] as disjoint completeness and S-completeness with further restriction of order boundedness for the disjoint set and replacing singleton set by an arbitrary subset in a Riesz space respectively. Infact, these completeness properties of a Riesz space are weaker than that of order completeness. Indeed, it was proved in [228] that a Riesz space is order complete if and only if it is Archimedean, disjoint complete and relatively uniformly complete; whereas in [226], Veksler proved that the S-completion

of the space C(X), where X is a compact Hausdorff space, is order complete if and only if it is a Banach space. Using the results of Veksler and Geiler [228], Bernau [17] proved that the lateral and order completion of an Archimedean lattice group commute with each other.

It is clear from the above diagram that the importance of an order complete Riesz space containing a band B lies in the expression of each element of the space uniquely as the sum of two disjoint elements, one from the band B and the other one from $B^{\perp}$. However, the bands B and $B^{\perp}$ are not necessarily principal bands. Prior to the implications of Diagram 2.1.1, Pinsker [184, 185] around the year 1945, proved

(*) "Every order complete Riesz space is the direct sum of a complete family of mutually disjoint principal bands."

This result has significance in the proof of another result due to Pinsker, which embeds an order complete Riesz space as a complete ideal of an order complete Riesz space with unit, thereby reducing the study of an arbitrary Riesz space to the study of a Riesz space with unit, where by a unit we mean an element of a Riesz space X which generates X as its band.

Although, Pinsker's result (*) guarantees the existence of an order decomposition (cf.Definition 1.2.17) in an order complete Riesz space, which in turn is useful in dealing with the projection operators from the space into the bands forming the decomposition; it is not always easy to deal with the properties related to the convergence, representation of elements of the space in terms of the elements of the bands etc. However, countable restriction on the indexing set of the order

decomposition led M.Gupta and the author to introduce many new concepts like similarity of order decomposition, associated o-sequence spaces etc.. Further, the study of countable order decomposition is justified in view of examples of Riesz spaces having countable decomposition. Also, the sequential order dual of a Riesz space having countable order decomposition possesses such an order decomposition. All these results form the material of the last chapter of this dissertation.

2.3 Locally Solid Riesz Spaces: These spaces are generalizations of normed lattices which are Riesz spaces equipped with the Riesz norm; cf. Definition 1.3.7 . The importance of studying vector lattices in connection with the Banach theory of normed vector spaces was first recognized by Kantorovich and his group of pupils, namely, B.Z.Vulikh , A.G.Pinsker, A.I.Yudin and others in the late thirties and the beginning of forties [105],[229]. This study was later picked up by Luxemburg and Zaanen who contributed extensively to the growth of this theory through a series of papers entitled 'Notes on Banach function spaces' ; cf. [132,(i),(ii),(iii),(iv), (v),(vi),(vii),(viii),(ix),(x), (xi), (xii), (xiii)]. However, a systematic account of the theory of Banach lattices is also available now in the monographs [36], [110] and [202].

Almost contemporary to Kantorovich, following a direction different from the one adopted by him, M.G.Krein [118, 119] initiated the task of ordering Banach spaces by considering a set of positive elements, which he called a cone, and developed the theory of partially ordered Banach spaces that are not necessarily vector lattices. Continuing the investigations on the

impact of order properties of a partially ordered Banach space on its dual , around the year 1939 Krein joined by J.Grosberg [120] proved

(*) " A continuous linear functional f on an ordered normed space with cone K admits a decomposition C (i.e. ,$f=f_1-f_2$, f_1,f_2 are continuous linear functionals on X with $||f_1|| + ||f_2|| \leq C||f||$) if and only if the cone K satisfies the condition C (i.e., $||x||\leq C$ whenever $y\leq x\leq z$, $||y||\leq 1, ||z||\leq 1$)."

Almost nine years later, Krein along with M.A.Rutman published a long paper [121] dealing with the investigations on continuous linear operators defined on such spaces . Indeed, they introduced the concept of normal and minihedral cones; where by a normal cone they meant a cone for which there exists $\delta >0$ such that $x_i\in$ K and $||x_i||=1$, $i=1,2$, imply that $||x_1+x_2|| > \delta$, a notion equivalent to the semimonotonic character of the norm,i.e.,for some constant $L >0$, $||x_1|| \leq L||x_2||$ whenever $x_1,x_2 \in K$ with $x_1\leq x_2$, cf.[116]. Using these concepts, they rephrased the theorem of Kakutani on L-space in the following form:

"A Banach space B is isomorphic to the space of continuous functions on a compact Hausdorff space if and only if there exists in B a normal minihedral cone with non empty interior. "

Besides, they also proved results on the common fixed points and the characteristic vectors of an abelian family of positive operators and their adjoints; and in the case of completely continuous operators , they considered a number of properties related to their strong positiveness (an operator T on B is said to be strong positive if for each x in K, there exists n in $\mathbb{N}$

with $T^n(x) \geq \theta$), decompositions and spectrum; for instance, we may quote

"Let T be a positive completely continuous operator on a partially ordered Banach space B such that its spectrum S contains a non zero point. Then there exist a positive characteristic number ρ greater than the absolute value of each member in S and vectors v in K and ϕ in $\tilde{K}$ such that $T(v)=\rho v$ and $T^*(\phi)=\rho\phi$."

On the other hand, in the later part of fifties Bonsall [26], I.Namioka [160], J.Kist [111] and D.Weston [232] got interested in generalizing the work of Krein on ordered normed spaces to ordered locally convex spaces. Their main aim was to obtain conditions under which any continuous linear functional on an ordered locally convex space can be written as the difference of two positive continuous linear functionals. In this direction, Bonsall [26] generalized the Grosberg and Krein's theorem as quoted in (*), in the following form :

" For a Hausdorff locally convex space X, the following statements are equivalent:

(a) For each neighbourhood U of θ, there exists another neighbourhood V of θ such that every linear functional f with $|f(x)| \leq 1$, $\forall\ x \in U$, satisfies $f = f_1 - f_2$, where f_1 and f_2 are two positive continuous linear functionals with $|f_i(x)| \leq 1$, $\forall\ x \in V$, $i=1,2$.

(b) For each neighbourhood U of θ, there exists another neighbourhood V of θ such that $x \in U$ whenever $y \leq x \leq z$, with y,z in V."

Let us note that the condition (a) above is a different

version of the statement that any continuous linear functional on X is the difference of two positive continuous linear functionals.

Namioka [160] who termed an OTVS with a normal cone (cf. Definition 1.3.5) as locally full , proved

"A linear functional f on an ordered locally convex space (X,τ) is dominated by a τ-continuous linear functional on X if and only if there is a τ-neighbourhood U of θ such that $f(y) \leq 1$ for each positive y with $y \leq x$ and $x \in U$."

This, in turn, yields

(+) "A continuous linear functional on an ordered locally full, locally convex space, is the difference of two positive continuous linear functionals."

Kist [111] termed a locally full, locally convex space, a *locally o-convex space* and proved (+) in the form as given above. This result was also proved independently by Weston [232] in the year 1957.

Though the mathematicians had started working on the ordered locally convex spaces, the researches on ordered normed spaces were also going on simultaneously. In the process of considering the converse problem of Krein and Grosberg stated in (*), T.Andô [11], D.A.Edwards [49] and A.J.Ellis [51] independently published papers dealing with the same. Whereas, Andô concentrated on finding conditions on the partially ordered Banach space B and its dual B^* so that the cone and its dual cone become generating for the spaces; Edwards and Ellis made use of the base for the cone and the order interval $[\theta,e]$, where e is the strong unit of the space, in order to make it the dual of

some Banach space.

In all the above results related to the decomposition of a continuous linear functional as the difference of two positive continuous linear functionals, it was observed that the normality of the cone plays a crucial role; indeed, this concept relates the order structure of the space with its topological structure and makes the dual cone generating. However, the generating character of the cone led Bonsall [23] to introduce the notion of bounded decomposition property (i.e., for each $x \in X$, there exist bounded sequences $\{y_k\}$ and $\{z_k\}$ in K such that $\{y_k - z_k\}$ norm converges to x), which becomes equivalent to the fact that the cone K of an ordered Banach space is a b-cone ; cf. [175],p 68. Generalizing the concepts of b- and strict b-cones to the study of locally convex spaces, Schaefer [202] who termed the space with a strict b-cone as the space having BZ-property , proved the following duality relation between normal and ζ- cones (cf. Definition 1.3.5):

" Let <X,Y> be a dual pair of vector spaces X and Y ordered respectively by the cones K and $\tilde{K}$. If $\tilde{K}$ is an ζ - cone in Y for a saturated class ζ of $\sigma(Y,X)$-bounded sets, then K is normal for the corresponding ζ -topology on X; conversely if K is normal for an ζ -topology on X consistent with the dual pair <X,Y>, then $\tilde{K}$ is a strict ζ -cone in Y."

Around the year 1962, Andô [11] proved the equivalence of the generating character of the closed cone with its b- and strict b- nature in a partially ordered Banach space . However, relating the normality of the dual cone $\tilde{K}$ with the strict b-character of K, he proved that the cone in an ordered Banach

space is a strict b-cone if the dual cone is normal. On the other hand, J.Riedl in his long paper [192] which forms a part of his doctoral dissertation, concentrated on various aspects of ordered locally convex spaces, especially the characterizations of normal cones, representation of spaces with normal cones as subspaces of C(X) and the extensions of positive linear mappings from subspaces to the whole space. Indeed, he proved :

" An ordered locally convex space X is isomorphic in both the order and topological sense to a subspace of C(X) for some locally compact space X, if and only if the positive cone of X is closed and normal."

" If a linear subspace H of a locally convex space X ordered by a cone K, contains an interior point of K, then every positive linear mapping from H into an order complete TVS Y with a normal cone, can be extended to a positive continuous linear mapping from X into Y."

Almost a decade later, Kung-Fu Ng and M.Duhoux jointly published a paper [164] dealing with the problem of Krein and Grosberg by using the duality theory of locally convex spaces and exhibited the duality relationship between order-convex hull or full hull and decomposable kernels of zero neighbourhoods in the space. Infact, there are a number of papers relating various properties of an OTVS with different types of cones and vice-versa, and it is difficult to cite the contributions of all; however, we may mention a few names of mathematicians, for instance, I.I.Ĉuĉaev [34], V.A.Geiler, I.F.Danilenko and Ĉuĉaev [62, 63] and Thamelt Wolfgang [217].

In an OTVS with a normal cone, every order bounded set

is topologically bounded; but the converse is not necessarily true e.g. the set $\{e^n: n=1,2,\ldots\}$ in ℓ^1 with the usual ordering is bounded in ℓ^1-norm, but it is not order bounded in ℓ^1. Researches related to the both way implications of topological boundedness to order boundedness and vice-versa, have been carried out by A.V.Mironov and B.T. Sarymsakov [145], D.D. Šulǵa [213] and others.

After the discovery of Banach lattices by Kantorovich, the main two directions with which mathematicians working in this field were occupied, were the theory of Banach or locally convex spaces ordered by different types of cones and the study of various linear topologies on Riesz spaces. We have already seen in the preceding paragraphs a brief chronological development of the first aspect of this theory. We now take up the second aspect of this generalization. Indeed, it was G.T.Roberts [195] who around the year 1952, initiated topologizing a Riesz space with the help of neighbourhood system in such a way that the lattice operations become continuous with respect to the topology. He further introduced the concept of the absolute envelope (i.e.,solid hull) of a set in order to define self absolutely enveloped (i.e., locally solid) topologies and proved the continuity of the lattice operations for such topologies. Besides, he also considered the solid polar topology on X for the dual pair $\langle X,X^b\rangle$ and proved that X^* is a solid subspace of X^b. Around 1956, C.Goffman [66] dealt with the seminorms compatible with the ordering structure of a Riesz space, which in our terminology are Riesz seminorms, and proved the existence of the finest locally convex topology generated by such seminorms,

which in certain known function spaces , namely, C(X) and L^p, $p\geq 1$, coincides with the norm topology. Besides, the contributions of I.Kawai [108], Namoika [160], Wong [237, 238], Ng [162], Burkinshaw and P.G.Dodds [28], in the direction of locally convex space and locally solid topology, available in the monographs [6],[55],[175],[202],[243], are also worth mentioning.

In this dissertation, we confine our study to those OVS or Riesz spaces which are order complete, the importance of this concept already being mentioned in the preceding section. While combining the linear topological structure with the order structure it is natural to inquire the relationship between order completeness and the topological completeness. An attempt in this direction was successfully made by Nakano [158] in the year 1952, who proved the topological completeness of a locally solid Riesz space, equipped with the topology having a neighbourhood system at origin consisting of solid, order complete sets (i.e. locally order complete space) such that each topologically bounded set has supremum (boundedly order complete space); in other words , every locally order complete and boundedly order complete locally solid Riesz space is complete, cf. [175],p.142. Removing the vector lattice structure and restricting the topology , Schaefer [201] proved a generalization of Amemiya's result [10] in the following form :

"A metrizable ordered locally convex space (X,τ) with a normal b-cone is complete if each monotonically increasing τ-bounded sequence has a supremum."

Around the year 1972, Ng [163] proved several equivalent conditions for the completeness of a partially ordered

metrizable locally convex space X with normal generating cone, and derived the following result which is an improvement of the above result:

"A metrizable locally convex space (X,τ) ordered by a normal b-cone, is complete if each τ-bounded subset of X is ℓ^1-*summable*, i.e. for each positive sequence $\{x_n\}$ in the set, the sequence $\{\sum_{i=1}^{n} \lambda_i x_i \ : n \geq 1\}$ where $\{ \lambda_n\} \in \ell^1$ with $\lambda_n \geq 0$, has a supremum in X."

Almost simultaneously, G.J.O.Jameson [90] and Wong [239] respectively established sufficient conditions for deriving the topological completeness and the monotonically sequential completeness (i.e., every increasing Cauchy sequence has a limit in the topology) of a metrizable ordered locally convex space from its fundamentally σ-order completeness (i.e., each increasing Cauchy sequence has a supremum).

The literature of this subject matter is too vast, to survey all the results of mathematicians who contributed to this theory, however, our bibliography would be incomplete if we don't mention the names of Duhoux [45], T.Nishiura [167], J.F.Aarnes [1], Fremlin [54], Aliprantis [3], V.A.Solovev [211, 212], G.Ja Lozanovskiî [129], R.Reichard [191], A.V.Potepun [188] etc.

Finally, in this section we restrict ourselves to various order topologies which may be defined on an OVS or for dual pairs of ordered vector spaces; for instance, we consider order topology or order bound topology τ_o and the topologies $o(X,Y)$, $\tau_s(X,Y)$, $\tau_s(X,Y)^*$ defined corresponding to a dual pair $\langle X,Y \rangle$ of ordered vector spaces X and Y.

The order topology τ_o was studied independently by

Namioka [160] and Schaefer [201] in the late fifties. Whereas, Namioka introduced this topology as the locally convex topology on an OVS X with a neighbourhood basis consisting of balanced, convex sets which absorb order bounded subsets of X, Schaefer defined it as the finest locally convex topology on an OVS for which any order bounded set is topologically bounded; the equivalence of the two statements being observed by both of them. Various results related to this topology are contained in [202] and [243]; for instance, Schaefer [202],p.231, proved that the order toplogy of an Archimedean OVS with a strong unit, is the normed topology for which the cone is normal. Wong [242] considered this topology on bornological Riesz spaces and proved its locally solid character under the restriction that the solid hull of each topologically bounded set is topologically bounded. He further investigated several equivalent conditions for the topology of a bornological Riesz space to coincide with τ_o. Keim Dieter [109] studied the order topology of the direct sum and product of a family of Riesz spaces equipped with order topologies. Recently, V.Murali [151] has considered the linear order bound topology which is defined to be the finest linear topology for which any order bounded set is topologically bounded and has proved results analogous to τ_o.

The topology o(X,Y) defined corresponding to a dual pair <X,Y> of ordered vector spaces, as the topology of uniform convergence on order bounded sets in Y, was introduced by Peressini in the year 1961 [172].This topology coincides with the Riesz space weak topology (absolute weak topology) considered in 1960 by Gordon [68] for a Riesz space X, where Y is the set of

positive linear functionals on X; cf. the paragraph preceding Proposition 1.3.14. Gordon [68] proved several results related to this topology and cited two examples involving C[0,1] such that in one case the weak topology $\sigma(X,Y)$ coincides with $o(X,Y)$ and in the other, the two are different. A.Pietsch [182] considered this topology for his investigations on summability in Riesz spaces. Three years later Komura-Koshi [113] proved that the topology of a nuclear locally solid Riesz space becomes discrete if it is also complete; (X is said to be discrete if it has a complete system of disjoint discrete elements, i.e., elements for which the ideals generated by them are the vector subspace generated by them). They have made use of their results for proving a characterization of nuclearity of a locally convex solid Riesz space in terms of a perfect space Λ equipped with $o(\Lambda,\Lambda^*)$. Motivated by Komura-Koshi's work, B.Walsh [230] derived necessary and sufficient conditions for a locally convex lattice to be representable as a normal Köthe space. Around 1969, J.T.Marti and D.R.Sherbert [142] considered Schauder bases (cf. [97, 98] for the detailed theory of Schauder bases in locally convex spaces) in an OTVS (X,τ) equipped with $o(X,X^*)$. On the other hand Irina Popa [187] dealt with the compactness of sets in a Banach lattice X for the topology $o(X,X^*)$.

Coming to the discussion of order polar topologies defined for a dual pair $\langle X,Y\rangle$ of Riesz spaces where Y is an ideal in X^b, we begin with the work of M.Duhoux [46] who introduced the topologies $\tau_s(X,Y)$ and $\tau(X_s,Y)^*$ as the topologies of uniform convergence on all convex, solid , $\sigma(Y,X)$-relatively compact subsets of Y and convex, solid, $\sigma(Y,X)$-relatively compact subsets

of Y satisfying A_3 respectively and proved

"$\tau_s(X,Y)$ (resp. $\tau_s(X,Y)^*$) is the finest locally convex solid topology (resp. the finest locally convex solid topology satisfying A_3) on Y which is coarser than the Mackey topology $\tau(X,Y)$ and admits Y as its dual."

He made use of this result for carrying out his investigation on o-weakly compact mappings ; cf. also Section 5 of this chapter.

It would be appropriate to point out here that the topologies with which we have come across during our survey of results on this subject, are the ones discussed above; though it is possible to have a number of locally convex polar topologies defined corresponding to a dual pair. We therefore, look for more locally convex topologies which arise from the mixed structure of order and dual relationship of ordered vector spaces or Riesz spaces. In Chapter 5 of this thesis, we have succeeded in obtaining some more polar topologies on a Riesz space X, namely, $\tau_c(X,Y)$, $\tau_{so}(X,Y)$ and $T_{op}(X^*,X)$.

2.4 Vector Valued Sequence Spaces: The theory of generalized or vector valued sequence spaces (VVSS), which has emerged as a natural generalization of scalar valued sequence spaces (SVSS), poses a natural question regarding the impact of ordering of the underlying space X on $\Lambda(X)$ and vice-versa, in case X is assumed to be an ordered vector space. Before giving the salient features of the work done in this direction, we sketch in this section a brief account of various contributions made in this theory eversince its growth as generalization of SVSS. However, Chapter 1 of doctoral dissertations of J.Patterson [170] and K.L.N.Rao [190] are also devoted to the historical development of this

theory and for this purpose we refer to these as well.

As mentioned in [170], the contributions of I.M.Gelfand [64] and R.S.Phillips [177] on some abstractly valued function spaces, are the source of motivation for studying VVSS. Besides, A.Grothendieck [74] made use of tensor product of SVSS ℓ^p with a Banach space X for his work on the theory of nuclear spaces. Also, Grothendieck's [74] and Schwartz's [210] generalizations of holomorphic mappings and distributions respectively from scalar valued to vector valued case are equally responsible for the introduction of VVSS. Certain type of VVSS were also used by C.Roumieu [199] in 1960 for his study on generalized distributions. However, it was A.Pietsch [181] who observed the importance of generalized perfect sequence spaces in the study of linear operators from a perfect SVSS to a locally convex space and presented a systematic study of VVSS of the type $\lambda(X)$ defined with the help of a SVSS λ and a locally convex space X as follows

$$\lambda(X) = \{ \{x_n\} \subset X: \{f(x_i)\} \in \lambda, \text{ for all } f \in X^* \}.$$

He studied various properties related to the characterizations of boundedness, compactness of subsets of $\lambda(X)$ after topologizing these spaces with the help of topology of X and λ.

The study of VVSS as generalization of Köthe and Toeplitz theory [115] of SVSS (cf. [96] and [114] also), was carried out by N.Phŏung Cǎc [178] who defined $\Lambda(X)$ as a vector space of sequences from a vector space X, with respect to the co-ordinate wise operations. For a dual pair ⟨X,Y⟩ of vector spaces, he introduced the generalized Köthe dual $\Lambda^x(Y)$ of $\Lambda(X)$ defined as in Chapter 1 and obtained many results for VVSS, analogues of which in SVSS are available in [96],[114]. He

further continued his investigations on VVSS in his later work [179,180], where he respectively expressed the topological duals of certain VVSS as generalized Köthe duals; and some spaces of functions and their duals, as the generalized sequence spaces and their Köthe duals.

Almost simultaneously but independently, D.A.Gregory [72], De-Grande De-Kimpe [37] and R.C.Rosier [198] submitted their doctoral theses on vector valued sequence spaces, where the space $\lambda(X)$ as defined in (1.4.11) was considered by all of them; though the overlapping of the material contained in these dissertations is very little. Indeed, Gregory worked on the lines of Căc for the dual pair $\langle\lambda(X),\mu(Y)\rangle$, where $\mu(Y)$ is a subspace of $\Lambda^{x}(Y)$ with $\phi(Y) \subset \mu(Y)$, and after introducing the notion of solid topology (normal ζ-topology, cf. Section 1.4), he proved that the Mackey and strong topologies are solid topologies under some restrictions. He also considered matrix transformations on the space $\lambda(X)$, besides proving some properties of the β-dual of the space $\lambda(X)$. De-Grande De-Kimpe defined $\lambda\{X\}$ as

$$\lambda\{X\}= \{ \{x_n\}: x_n \in X,\ n\geq 1,\ \{ \sup_{a\in M} |\langle x_n,a\rangle| \} \in \lambda,\ \forall\ M \in \mathcal{M} \},$$

where X is a Hausdorff locally convex space, λ is a perfect SVSS equipped with a solid topology and $\mathcal{M}$ is the family of all equicontinuous subsets of the dual space X^* of X. She topologized the space $\lambda\{X\}$ with the help of the topologies of λ and X and proved that the topological dual of the space always contains sequences from the strong dual of X and it is precisely $\lambda^{x}(X^*)$ for a normed space X. She also considered the topological tensor product representation of the space $\lambda\{X\}$, characterized the nuclearity of $\lambda\{X\}$ in terms of the nuclearity of λ and X and

proved the results on operators of Λ-type. On the other hand, Rosier equipped the space $\lambda\{X\}$ as defined by Gregory and De-Grande De-Kimpe, with two locally convex topologies (called the m-topologies) generated by the families $\{\Pi_{M,U}: M \in \mathcal{M}, U \in \mathcal{U}\}$ and $\{\varepsilon_{M,U}: M \in \mathcal{M}, U \in \mathcal{U}\}$ of seminorms defined as

$$\Pi_{M,U}(\{x_i\}) = \sup_{\{\alpha_i\}\in M} \sum_{i\geq 1} |\alpha_i| \, p_U(x_i) ,$$

$$\varepsilon_{M,U}(\{x_i\}) = \sup_{\substack{\{\alpha_i\}\in M \\ f\in U^\circ}} \sum_{i\geq 1} |\alpha_i| \, |f(x_i)|,$$

where $\mathcal{U}$ is the fundamental neighbourhood system at origin and $\mathcal{M}$ is a certain class of bounded sets in λ^x covering λ^x. He also investigated the dual of the subspace $[\lambda\{X\}_M]$ consisting of those elements of $\lambda\{X\}$, which are limits of their sections, as the space of all vectors $\bar{a}=\{a_n\}$, $a_n \in X^*$, which have representations of the forms $\bar{a} = \bar{\alpha}\bar{f} = \{\alpha_n f_n\}$ with $\bar{\alpha} \in \lambda^x$ and $\bar{f}=\{f_n\}$, an equicontinuous subset of X^*.

Almost three years later, M.Gupta, P.K.Kamthan and K.L.N. Rao [83, 84, 85] took up the study of duality relationships between various VVSS and characterized weakly bounded sets in such spaces. They also discovered the applications of VVSS in relation to Schauder decomposition (a concept more general than the Schauder bases in a locally convex spaces) and vice-versa, (let us recall that a sequence $\{M_i\}$ of subspaces in a locally convex space X is a Schauder decomposition for X if each $x \in X$ is uniquely expressed as $x = \sum_{i\geq 1} x_i$, $x_i \in M_i$, $i\geq 1$ and the projections $P_i : X \longrightarrow M_i$ given by $P_i(x) = x_i$, $i \geq 1$ are continuous); for instance, they proved that every VVSS $\Lambda(X)$ with $\phi(X) \subset \Lambda(X)$, where X is a locally convex space , has a Schauder

decomposition $\{N_i\}$ with respect to the weak topology $\sigma(\Lambda(X),\Lambda^X(X^*))$, where $N_i = \{\delta_i^x : x\in X\}$, $\forall$ $i\geq 1$; and if $\{N_i\}$ is e-Schauder (i.e., the sequence $\{Q_n\}$, where $Q_n(x) = \sum_{i=1}^{n} P_i(x)$, $x\in X$, is equicontinuous on X), then $\Lambda^X(X^*) = \phi(X^*)$.

Around 1980, Gupta, Kamthan, and Patterson [82] succeeded in characterizing the topological dual of a barrelled VVSS as a generalized Köthe β-dual. Besides, Gupta and Patterson considered problems related to the characterizations of compact sets in VVSS, S-Radon Riesz property in $\ell^p(X)$ and matrix transformations on VVSS $\Lambda(X)$, cf. [75,76,77,78]. Indeed in [78], they tackled the problem of characterizing the nuclearity and precompactness of diagonal operators.

On the other hand, N.R. Das and Gupta who were carrying out investigations on mixed structure of locally convex spaces [80] observed that the VVSS which possesses a Schauder decomposition in a natural way, are useful in giving the γ-completion of a bi-locally convex space , cf. [79],p.187. They along with Kamthan also considered the mixed structure of VVSS in [81].

The theory of VVSS has also been enriched by the contributions of I.E.Leonard [124], Ju.I.Grabinov [70, 71], H.G.Garnir [65], L.b.Patricia [171], M.Florencio, P.J.Paul [53] and others. However, the problem related to the generalization of summability domains from scalar case to vector case was initiated by A.Robinson [197] in 1950, who characterized the matrix transformations on certain VVSS formed by the elements of a Banach space. This has further been carried out by I.J.Maddox [137,138], B.Thorpe [218] and others. Still the researches in

this direction are going on by considering locally convex spaces in place of normed spaces and the process of advancing this theory by more and more contributions is on.

The real SVSS are OVS in a natural way with respect to the co-ordinate wise ordering. Indeed, motivated by the work of Dieudonné [40] on function spaces which are clearly partially ordered spaces, Peressini tried to relate the ordering and topological structure of SVSS in [173], and mentioned modifications of some of the results in order to apply them for function spaces. For instance, he proved:

" The lattice operations on a Riesz subspace λ of w, containing ϕ, are $\sigma(\lambda^{x},\lambda)$-continuous if and only if $\lambda^{x} = \phi$, or equivalently, $\sigma(\lambda,\lambda^{x})$ is metrizable."

Equipping λ with the topology $o(\lambda,\lambda^{x})$ and calling a linear operator bounded if it maps neighbourhoods of zero into bounded sets, he characterized the boundedness of a positive operator in the form:

"A positive linear operator from λ into itself is $o(\lambda,\lambda^{x})$- bounded if and only if it is $\sigma(\lambda,\lambda^{x})$-$\sigma(\lambda,\lambda^{x})$ continuous and λ^{x} contains an element which is an order unit for the range of the adjoint operator T^{*}."

He continued his investigations related to the order structure of the class of linear mappings defined on sequence spaces in [174]. Denoting the space of all $\sigma(\lambda,\lambda^{x})$-$\sigma(\mu,\mu^{x})$ continuous linear maps from a sequence space λ to another sequence space μ, by $\mathcal{L}_m(\lambda,\mu)$ which is, indeed, the class of matrix transformations from λ to μ in view of the result of Köthe and Toeplitz [115] (cf. also [96],p.205), he proved :

"If $\mathcal{L}_m(\lambda,\mu)$ is a lattice, then μ is a lattice and each $f\in\mathcal{L}_m(\lambda,\mu)$ is $o(\lambda,\lambda^x)-o(\mu,\mu^x)$ continuous. Further, if μ is normal, $\mathcal{L}_m(\lambda,\mu)$ is an ideal of $\mathcal{L}^b(\lambda,\mu)$."

Two years later, Peressini and Sherbert jointly published a paper [176] in which they established many interesting properties related to the ordering of a sequence space; for example they proved that a sequence space is always order separable and every perfect sequence space has boundedness property. They used some of these results in investigating the conditions for the class $\mathcal{L}^{so}(\lambda,\mu)$ to coincide with $\mathcal{L}_m(\lambda,\mu)$.

The study of OVVSS defined over an arbitrary index set I and by the elements of an OTVS X ordered by a closed normal cone K, was initiated by B.Walsh [231] around the year 1973. Denoting by $\mathcal{F}(I)$, the *class of finite subsets of I*, he defined the cones in the product space X^I:

$m_\infty^+(I,X)=\{\{x_i\}_{i\in I}: x_i\in K$ and $x_i\le y$, for some $y\in K$ and for each $i\in I\}$;

$a_o^+(I,X)=\{\{x_i\}_{i\in I}$: $x_i\in K, \forall\ i\in I$ and the net $\{\sum_{i\in S} x_i : S\in\mathcal{F}(I)\}$ is Cauchy in $X\}$;

$b_1^+(I,X)=\{\{x_i\}_{i\in I}$: $x_i\in K, \forall\ i\in I$ and $\sum_{i\in S} x_i\le y$, for some $y\in K$ and for every S in $\mathcal{F}(I)\}$;

$a_1^+(I,X)=\{\{x_i\}_{i\in I}$: $x_i\in K, \forall\ i\in I$ and $\{\sum_{i\in S} x_i : S\in\mathcal{F}(I)\}$ is topologically bounded in $X\}$.

He used the notations $m_\infty(I,X)$, $a_o(I,X)$, $b_1(I,X)$ and $a_1(I,X)$ for the spaces generated by the above cones respectively and after topologizing these spaces appropriately, he proved many interesting results dealing with their topological properties and the duality relationships. For instance, he proved

" If (X,τ) is a Fréchet space with the topology τ generated by the family $\mathfrak{D}=\{q_n;\ n\ge 1\}$ of monotone seminorms, then the spaces $(m_\infty(I,X),\mathcal{T}_1)$, $(a_o(I,X),\mathcal{T}_2)$, and $(a_1(I,X),\mathcal{T}_3)$ are

also Fréchét, where the topologies $\mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{T}_3$ are generated by the families $\{q_\infty : q \in \mathcal{D}\}$, $\{q_o : q \in \mathcal{D}\}$ and $\{q_1 : q \in \mathcal{D}\}$ of monotone seminorms given by

$$q_\infty(\bar{x}) = \inf \{ \max \{q(y), q(z)\} : y, z \in K, -y \leq x_i \leq z, \forall\ i \in I \},$$

$$q_o(\bar{x}) = \inf \{ \Pi_q(\{x_i^+\}) - \Pi_q(\{x_i^-\}) : \bar{x} = \{x_i\} = \{x_i^+\} - \{x_i^-\}\}$$

$$= q_1(\bar{x}),$$

where for $\{y_i\}$ in $a_j^+(I,X)$, $j=0,1$, $\Pi_q(\{y_i\}) = \sup_{S \in \mathcal{F}(I)} \{q(\sum_{i \in S} y_i)\} < \infty$." . He applied these results on VVSS for generalizing the results of Schlotterbeck [208] and Schaefer [204] on K-absolutely-summing and majorizing operators from Banach lattices to locally convex spaces.

Almost at the same time, R. Cristescu [32] considered directed subspaces $\Lambda(X)$ of $\Omega(X)$ containing $\phi(X)$ for an OVS X and investigated conditions for $\Lambda(X)$ to inherit various lattice properties possessed by X. In 1979, D.Tofan published a paper [219] dealing with the VVSS $\Lambda(X)$ containing $\phi(X)$ for a vector space X, and characterized the equality of $\Lambda(X)$ with $\Omega(X)$. For a Riesz subspace $\Lambda(X)$ of $\Omega(X)$, he also considered results related to the unity and boundedness property.

Though much has been said about the advancement of the theory of VVSS, yet much is left to be accomplished; for instance, one may look for finding the order structure of the space of matrix transformation from an OVVSS $\Lambda(X)$ to another OVVSS $\mu(Y)$ and several order properties of the space $\Lambda(X)$, for example diagonal property and various other problems related to the order topologies on such spaces. The task is not easy and at the same time, success is not always possible; however attempts are made. This thesis is one such effort in this direction, where

we have tried to investigate the Köthe dual $\Lambda^{x}(X^{so})$ of $\Lambda(X)$ corresponding to the dual pair $\langle X,X^{so}\rangle$ of a Riesz space X and its sequential order dual X^{so}; besides tackling problems concerning the matrix representation of sequentially order continuous linear maps on VVSS, and considering the duality of certain OVVSS in Chapters 3 and 4.

2.5 Linear Operators On Riesz Spaces : This section sketches a brief history of certain subspaces of the class of linear operators defined on Riesz spaces; indeed, these are the class of order bounded operators , order continuous operators, Riesz homomorphisms and compact operators if the topologies are also assigned to the Riesz spaces.

The order bounded operators on OVS, which map order bounded sets into order bounded sets, were essentially introduced by Riesz [194], as the difference of two positive linear functionals, referred to regular operators by Kantorovich [104] and Vulikh [229]. Indeed, in [104], Kantorovich proved that the regular linear operators from an arbitrary Riesz space X to an order complete Riesz space Y, are those maps which preserve order boundedness ; and the space consisting of all such operators is an order complete Riesz space. The latter result was earlier proved by Riesz [194] for the particular case when $Y=\mathbb{R}$.

On the other hand, Kantorovich [104] introduced several subclasses of the space $\mathfrak{L}(X,Y)$, with the help of convergence notions studied by him in [99, 103]. Indeed, using the notion of (s_{ℓ})- continuous operator for an operator which maps s-convergent sequence into ℓ-convergent sequence, he considered the classes H°_{t}, $H^{\circ}_{\circ}$,H^{t}_{t} and $H^{t}_{\circ}$ consisting of respectively $(\circ_{t})$, $(\circ_{\circ})$ (t_{t}) and

(t_o)- continuous operators and proved that the classes H^t_o and H^t_t are essentially the same and $H^o_t \subset H^o_o \subset H^t_t$. It would be worth mentioning here that by t-convergence he meant either a topological or (*)- convergence in the sense of ordering. Besides dealing with many results on the representation of operators belonging to these classes for general as well as particular domain and range spaces, he also proved that the pointwise order limit of a monotone sequence of regular operators is regular and is its order limit; a result given in Theorem 1.2.22 for nets. Most of these results are to be found in [229] in more generality.

Replacing sequences by nets, H.Nakano [156] developed the theory of order continuous operators on universally continuous and continuous semi-ordered spaces which are respectively the order complete and σ-order complete Riesz spaces in our terminology. Most of his results appeared in [156] have also been published in abridged form [159] by Wayne State University, Detroit in the year 1966. T.Ogasawara also considered the class of these maps and proved [168]

" The subspaces $\mathcal{L}^c(X,Y)$ and $\mathcal{L}^{so}(X,Y)$, consisting of respectively order bounded, order continuous and order bounded, σ-order continuous linear operators from a Riesz space X to an order complete Riesz space Y, are bands in the Riesz space $\mathcal{L}^b(X,Y)$ of all order bounded operators."

A year later, M.Nakamura characterized the order continuous linear functionals for some particular spaces in [152].

On the other hand, in the series of papers cited

earlier, Luxemburg and Zaanen called the order continuous and σ-order continuous linear functionals on Riesz spaces as normal integrals and integrals respectively; and proved that the class of singular functionals is a projection band in X^b, where by a singular functional they meant a member of the orthogonal complement of X^{so} in X^b; cf. [132(vi)],p 663. In 1967, Luxemburg [130] alone gave an example of an order complete Riesz space possessing an integral which is not normal, and determined conditions on a Riesz space so that every integral defined on this space becomes normal. Four years later, Luxemburg and Zaanen jointly published two papers [133, 134] dealing with the linear modulus $|T|$ of an order bounded operator T, where they characterized order bounded operators in terms of star continuity, and positive, negative and total variation of the operator T; besides relating their kernels. All the advancements made by these two mathematicians are available in their voluminous treatises [135] and [246].

During the decade of 1961 to 1970, R.G.Cogburn [31] applied order continuous linear operators in his study of conditional probability operators; whereas Mullins [147, 148] considered them on Köthe spaces and proved the equality of this class of operators defined on certain Riesz spaces which envelope the class of Köthe spaces, with the class consisting of linear operators continuous with respect the topology of uniform convergence on order intervals and the weak topology respectively.

In the process of extending the results of Kaplan [106] and Schaefer [203], O.Burkinshaw [27] introduced the concepts of

equi-l^1-continuity and equi-σ-continuity for the subsets of the bound and the sequential order duals of a Riesz space X and characterized their weak compactness. For instance, he proved:

" For a σ-order complete Riesz space X, a subset A of X^{so} is equi-ℓ^1-continuous $\Leftrightarrow$ it is equi-σ-continuous $\Leftrightarrow$ it is relatively $\sigma(X^{so},X)$ compact."

Around the year 1975, P.G.Dodds [42] motivated by the results of D.H.Fremlin and Burkinshaw on weak compactness , proved that for certain Riesz spaces X which are more general than order complete Riesz spaces , the pointwise limit of a sequence in X^b(resp. X^{so}or X^c) is a member of X^b(resp. X^{so}or X^c) and the pointwise convergence can be extended to the ideal generated by X in $(X^b)^c$.

The class of order bounded operators and its various subclasses, for instance, the class of extreme positive operators, family of discrete positive operators, collection of Riesz homomorphisms etc., defined on general as well as particular Riesz spaces have also been considered by Kaplan [107], A.J.Ellis [50], G. J.O.Jameson [89], C.D.Aliprantis [4], A.I.Veksler [223,225], C.T.Tucker [220, 221, 222] and others. Most of these results have already been appeared in the books [6], [55], [135], [175], [204] and [246].

Researches in the direction of enriching this theory on the lines of Banach space theory have been pursued by many mathematicians; for instance, a result concerning the extension of Riesz homomorphisms on the pattern of Hahn Banach theorem was proved by Z.Lepecki [126] in 1980. Relating the order structure of the range space with the equality of the classes of regular

and order bounded operators, Ju.A.Abramovič and V.A.Gejler [2] in the year 1982, answered in affirmative the question raised by Fremlin concerning the existence of a Riesz space which is not order complete but the space $\mathcal{L}^r(X,Y)$ of all regular operators from X to Y coincides with $\mathcal{L}^b(X,Y)$; and they further proved that a Riesz space Y would be order complete if $\mathcal{L}^r(X,Y)$ were a Riesz space and $\mathcal{L}^r(X,Y)=\mathcal{L}^b(X,Y)$, for every Riesz space X. In 1985, G.J.H.M.Buskes [29] investigated the conditions on the range space for the extension of *o- and c-continuous* Riesz homomorphisms from an ideal of the domain space to the whole space, where by an o- continuous (resp. c-continuous) Riesz homomorphism he meant a Riesz homomorphism ϕ defined on an ideal I of a Riesz space X to another Riesz space Y such that $\phi\{[\theta,x] \cap I\}$ for each $x \geq \theta$, is an order bounded subset in Y (resp. relative uniform convergence of null sequence in I is preserved under ϕ).

Around the year 1988, Schaefer [205] who defined order continuity of operators with the help of order convergence of filters (cf. [204], p.54), a notion stronger than that of order boundedness, gave a characterization for a regular operator to possess this property in terms of its adjoint mapping from Y^c into X^c, where X is a Riesz space and Y is an order complete Riesz space for which Y^c separates points of Y. Recent publications [214, 216] of C.Swartz, contain order versions of uniform boundedness principle and Banach-Steinhaus theorem in Riesz spaces.

Besides the above references, there are many more who have enriched this theory and we may cite some of them, for

example, [7], [13], [18], [144], [149], and [234]. We now begin our discussion on the developments of operators which are also assigned linear topologies besides being ordered . The discussion may be bifurcated in two ways - firstly consideration of continuous linear operators discussed above on OTVS and secondly the linear operators defined with the help of the mixed structure of ordering and topology of the space.

As mentioned in Section 2.3, the study of continuous linear functionals on an ordered normed space was initiated by Krein and Grosberg [120] followed by by Krein and Rutman [121], Bonsall [26], Namioka [160], Kist [111], Weston [232] and many others which have already been cited in this section. However, in this section , we would start with the work of Luxemburg and Zaanen who began their discussion on Riesz spaces with the paper [132(vi)], where they related the existence of a positive linear functional on a Riesz space with that of a Riesz seminorm and studied the properties of *Riesz annihilators* defined for subsets of a Riesz space and its order dual. They continued their investigations on annihilators for subsets of a normed Riesz space and its dual and also proved [132(vii)]

" The dual X^*, the space X_c^* consisting of all continuous integrals and the space X_s^* of all continuous singular functionals on a normed Riesz space X, are ideals in X^b; X^* is not necessarily a band in X^b and $X^* = X_c^* \oplus X_s^*$."

In fact, they investigated normed Riesz spaces from various aspects; for instance, they proved results concerning reflexivity, perfectness, order density character of various duals, separability etc., for such spaces. In short, a systematic

development of this theory owes a lot to their contributions.

Around the year 1970, G.Ja.Lozanovskii [128] characterized the order continuous linear functionals on the dual X^* of a normed Riesz space X as those members F of X^{**} for which $\sum F(f_\alpha)=0$ for any $\sigma(X^*,X^{**})$- absolutely summing family $\{f_\alpha:\alpha\in\Lambda\}$ in X^* such that $\sum f_\alpha(y) = 0$, for each $y \in X$; thereby showing that some order properties are uniquely determined by the norm.

Generalizing the result of P.M.Anselone [12] on ordered Banach spaces , R.L.James [91] proved:

" A pointwise limit T of a sequence $\{T_k\}$ of positive continuous linear operators from a barrelled OTVS X to a locally convex space Y ordered by a normal cone is continuous and the convergence is uniform on T-regular sets", ($A \subset X$ is a T-regular set if for each neighbourhood U of θ in Y and each x in A, there exist x^U, x_U in X such that $x^U\le x\le x_U$, and $T(x^U-x_U)\in U$ and the sets $A_U=\{ x_U: x\in A \}$, $A^U=\{ x^U: x\in A \}$ are totally bounded).

In the year 1974, D.Birnbaum [21] defined pre-regular maps from a Banach lattice X to another Banach lattice Y, as the difference of positive continuous linear maps from X into the bidual Y^{**} of Y and characterized the Banach lattices for which each continuous linear map is pre-regular. His investigations involved the concepts of absolutely summable, majorizing, absolutely majorizing operators which belong to the second category of our discussion to be taken up in the following paragraphs.

A major reference for contributions on linear operators of Banach lattices that incorporates the work of Schaefer and his school, namely, H.P. Lotz [127], R.J.Negal and U.Schlotterbeck

[161], M.Wolff [236] as well as the work of V.L.Levin [125], N.J.Neilson [166], Wickstead [233] and others, is the book by Schaefer himself [204]. Though the paper of B.Walsh [231] published in 1973 is to be found in the notes mentioned after Chapter 4 of [204], yet the material is not included in the texts ; for the results of Walsh mainly dealt with the generalizations of Schlotterbeck's work on Banach spaces to the setting of locally convex spaces. Besides, he also presented some counter-examples showing the indispensable character of Riesz space structure for some of Schlotterbeck's results; for instance, he gave a counter example showing that the space $\mathcal{L}^{\ell}(H,\ell^1)$ of all cone-absolutely summing operators from a Hilbert space H to the space ℓ^1, is not a Riesz space though this space coincides with the space of all continuous linear operators from H to ℓ^1; and Schlotterbeck's result states that the space $\mathcal{L}^{\ell}(X,Y)$ is a Banach lattice provided X and Y are Banach lattices and Y has property P; cf.[204],p.251. The locally convex version of this result was proved by Wong [241] in the following form:

" The space $\mathcal{L}^{\ell}(X,Y)$, where (X,τ) and (Y,S) are l.c.s.Riesz spaces such that Y is both locally and boundedly order complete, is an ideal in $\mathcal{L}^{b}(X,Y)$."

In the same paper, he also investigated conditions under which each continuous map on a l.c.s. Riesz space becomes majorizing. However, his earlier publication [240] of lecture notes contains several examples, counter examples and other contributions on cone pre-nuclear, precompact, nuclear, absolutely summing mappings.

On the other hand, compact linear mappings on normed

spaces which emerged as an outgrowth of Riesz's abstract theory of functional equations - a theory more general than the Fredholm theory of integral equations, have been investigated now in the setting of ordered Banach and locally convex spaces. Indeed, depending on the nature of domain and range spaces, there have been basically two directions of researches related with (i) the o-weakly compact operators defined from Riesz spaces to locally convex spaces , and (ii) the compact operators from l.c.s.Riesz spaces to l.c.s.Riesz spaces.

The class of o-weakly compact operators which map order intervals in a Riesz space X into relatively weakly compact sets in a Banach space Y, was introduced and characterized by P.G.Dodds [41] in the year 1975. Replacing Banach space X by a locally convex space and the order dual X^b by any band in it, M.Duhoux [46] generalized some of the Dodds' results by using a technique based on his work in [47], which was different from that of Dodds. For instance, he proved that a linear operator $T:X\longrightarrow Y$, mapping intervals of a Riesz space X into bounded sets with complete closure in a locally convex space Y, is o-weakly compact if and only if for every majorized increasing sequence $\{x_n\}$ in X, the sequence $\{T(x_n)\}$ is Cauchy in Y. He also established that the pointwise limit T of a sequence $\{T_n\}$ of $\tau(X,A)$-continuous, o-weakly compact mappings from an Archimedean Riesz space X having property I (cf. [42],also) to a quasi complete locally convex space Y is $\tau(X,A)$-continuous and o-weakly compact, where A is a band in Y^b.

On the other hand, in a seminar held at Bucharest in the year 1981, C.Niculescu [165] presented a paper dealing with

the characterizations of the operators of the type A and B introduced by him from a Banach lattice X to a Banach space Y as the operators T and S for which $\{T(x_n)\}$ (resp. $\{S(x_n)\}$) norm converges in Y whenever $\theta \le x_n \downarrow$ in X (resp. $\theta \le x_n \uparrow$, $||x_n|| \le M$, for some constant $M > 0$). Besides finding the relationship between operators of these two types, he proved implicitly the equivalence of the operators of type A with that of o-weakly compact operators in the process of characterizing these operators.

Almost four years later, Dodds [43], characterized the range of an order continuous, o-weakly compact mapping as zonoform; which is a certain type of convex and weakly compact set in a locally convex space.

Prior to the above publication, in the year 1979, Dodds along with Fremlin published a paper [44] dealing with the second aspect of our discussion; where they have tried to give a unified approach to the study of integral operators in various function spaces, made by Luxemburg and Zaanen [131] , J.J.Grobler [73], M.A.Krasnoselskii [117], Negal and Schlotterbeck [161] and Schaefer [204]. Indeed, they have introduced the concepts of L-weakly, PL-compact and M-weakly compact operators from a Banach lattice X to a Banach lattice Y, which respectively map norm bounded sets in X to L-weakly compact set (the bounded set A for which any disjoint positive sequence in the solid hull of A converges to zero), PL-compact set (the set B for which ϕ_g is relatively compact in $\hat{Y}_g$ for each positive g in Y^*, where ϕ_g is the quotient map from Y to the completion $\hat{Y}_g$ of $Y_g = Y \backslash N_g$, with respect to the norm $||\hat{z}|| = g(|z|)$, for $\hat{z} = z + N_g$, $N_g = \{z \in X$

$:g(|z|)=0\}$ and $z \in Y$) and carry norm bounded disjoint sequences in X into null sequences in Y. Another type of operators , defined from a Riesz space X to another Riesz space Y, mapping order intervals to $|\sigma|(Y,Y^b)$-precompact sets were termed by them as AMAL-compact operators. Depending heavily on the behaviour of disjoint sequences in a Riesz space, they proved several results interrelating these various concepts of operators amongst themselves as well as with that of compact operators . For instance, they proved

"Let X and Y be two Banach lattices such that norm for Y is order continuous. Then $T \in \mathcal{L}(X,Y)$ is compact if and only if it is L-weakly and PL-compact. In addition, if X^* has order continuous norm and T is order bounded, AMAL-compact operator, then T is compact $\Leftrightarrow$ T is L-weakly compact $\Leftrightarrow$ T is M-weakly compact $\Leftrightarrow$ $\lim_{n\to\infty} g_n(T(x_n))=0$, whenever $\{x_n\}$ and $\{g_n\}$ are positive disjoint norm bounded sequences in X and Y^* respectively."

Around the year 1981, inspired by the above work of Dodds and Fremlin , Michel Duhoux in a paper [48] that mainly deals with the order precompact sets, introduced o-precompact operators which are more general than AMAL-compact operators and are defined as operators that map order intervals of a l.c.s.Riesz space into precompact sets of another l.c.s. Riesz space. These operators in a natural way are special cases of precompact operators and the converse holds in the following form [48]:

" A continuous o-precompact linear mapping from a l.c.s. Riesz space (X,τ) to another l.c.s.Riesz space (Y,S) is precompact if $\beta(X^*,X) = o(X^*,X)$ on every τ-equicontinuous subset

of X^*."

He also observed that the result of Dodds and Fremlin on the collection of AMAL-compact mappings from a Riesz space X to another order complete Riesz space Y with $Y^b \subset Y^c$, in $\mathcal{L}^b(X,Y)$ (cf. also Theorem 3.4 of [44], p302) would be valid in the case of o-precompact operators; cf. Proposition 1.3.18 and [7].

The work of Schaefer [204], Negal and Schlotterbeck [161], Schep [207] and others on kernel operators in Banach function spaces, influenced Aliprantis, Burkinshaw and Duhoux [8] to consider abstract kernel operators on l.c.s.Riesz spaces. According to them an abstract kernel operator is a member of the band generated by the vector space of all continuous, finite rank operators from a l.c.s.Riesz space X to another l.c.s.Riesz space Y, which can be written as the difference of two positive weakly continuous operators from X to Y. Concerning the o-precompactness of an abstract kernel operator, they proved:

" let (X,τ) and (Y,S) be two l.c.s.Riesz spaces such that Y is order complete and the topologies S and $\beta(X^*,X)$ are Lebesgue topologies. Then a kernel operator T from X to Y is o-precompact if and only if $T^*(A)$ is $|\sigma|(X^*,X)$-precompact for each equicontinuous subset A of Y^* or equivalently T(B) is $|\sigma|(Y,Y^*)$- precompact for each τ-bounded subset B of Y ."

Generalizing further the results of Dodds and Fremlin on disjoint sequences in Banach lattices to the setting of l.c.s.Riesz spaces and using the above result, they derived several characterizations of the precompact kernel operators.

Having realized the nice order structure in certain spaces of compact or precompact operators, mathematicians got

interested in finding the nature of a positive operator and its power, when it is dominated by a compact or a precompact operator. It was Pitt [186] as well as Dodds and Fremlin [44] who respectively considered this problem for L_p spaces and Banach lattices in the year 1979. Making use of the lattice theoretic arguments, Dodds and Fremlin proved:

"Every $S \in [\theta,T] \subset \mathcal{L}^b(X,Y)$, where X and Y are two Banach lattices such that X^* and Y have order continuous norms, is compact provided T is compact."

Around 1982, H.Leinfelder [123] extended the above result in the form that any operator S from L_p to L_q such that $|S| \le \lambda|T|$, where T is a compact operator, can be approximated by a sequence of operators of the type $\sum_{i=1}^{n} R_i T S_i$, where $R_i : L_p \longrightarrow L_q$ and $S_i : L_p \longrightarrow L_p$, $\forall\ i \ge 1$, are norm bounded operators, in the operator norm. Another generalization of the above result for precompact operators on l.c.s.Riesz spaces was given by Cristescu [33] in the year 1984.

Recent contributions in developing the theory of linear operators on Riesz spaces made by C.Swartz [215], R.Murakami [150], Schaefer [206] and K.D.Schmidt [209] are also worthy of mention.

While dealing with the researches in vector valued sequence spaces, the idea of defining o-matrix transformations by utilizing the order convergence of the Riesz spaces, occurred to us. Having got the success in representing a sequentially order continuous linear map as an o-matrix transformation , it became natural for us to inquire about the various order properties of such a map in terms of the components operators and vice-versa. Besides, if the Riesz space and the OVVSS are equipped with

l.c.s. topologies, is it possible for us to go ahead with the study of o-matrix transformations when the mixed impact of ordering and topological properties are considered. An attempt in this direction has been made in Chapter 4 of this dissertation where we deal with the matrix representation of the adjoint of these maps as well as the characterization of o-precompactness of these maps, besides finding the conditions for these transformations to transform particular OVVSS amongst themselves. Though the study of o-matrix transformations has been initiated in this dissertation, an overview of various advancements made in the operator theory makes us feel that a lot can possibly be achieved as the wheel of time goes by.

CHAPTER 3

ORDERED GENERALIZED SEQUENCE SPACES

3.1 An Overview: This chapter essentially deals with the order as well as topological properties of a vector valued sequence space (VVSS) $\Lambda(X)$ defined by the elements of an ordered vector space or a Riesz space or a locally solid Riesz space X. Starting our discussion with some basic results involving the impact of ordering of the space X on a VVSS $\Lambda(X)$ and vice-versa in the next section; we study in Section 3.3, the topological properties of the space $\lambda(X)$ as defined in (1.4.11), where X is a locally solid Riesz space. The last section of this chapter, which has been divided into two subsections, incorporates results on the dual spaces of OVVSS. Whereas the first part is devoted to the identification of the class $[\Lambda(X)]^{so}$ of sequentially order continuous linear functionals on $\Lambda(X)$ with its generalized Köthe dual $\Lambda^{x}(X^{so})$; in the second subsection, we find the Köthe duals of particular OVVSS defined analogous to c_o, c, ℓ^1, ℓ^∞, bv and bv_o with the help of the order structure of an order complete Riesz space .

3.2 Order Properties of $\Lambda(X)$: Throughout this section, we write X to denote an ordered vector space (OVS) or a Riesz space. Let us also recall from Section 1.4, the VVSS $\Lambda(X)$ defined over a real vector space X, and the related notations .

If X is an OVS, we define the co-ordinatewise ordering in $\Lambda(X)$ as :

(3.2.1) $$\bar{x} \le \bar{y} \Leftrightarrow x_n \le y_n \;, \; \forall\, n \in \mathbb{N} \;,$$

for $\bar{x}$, $\bar{y}$ in $\Lambda(X)$. Then we have

Proposition 3.2.2: $\Lambda(X)$ is an OVS with the cone $\bar{K} = \{\bar{x} \in \Lambda(X) : x_n \in K, \forall n \geq 1\}$.

Proof: If X is an OVS, the ordering defined in $\Lambda(X)$ by (3.2.1) is clearly a partial order satisfying the conditions that $\bar{x}+\bar{z} \leq \bar{y}+\bar{z}$ and $\alpha\bar{x} \leq \alpha\bar{y}$ for all $\alpha \geq 0$, whenever $\bar{x} \leq \bar{y}$ and $\bar{x}, \bar{y}, \bar{z} \in \Lambda(X)$. It is also obvious that $\bar{K}$ is the set of all positive elements in $\Lambda(X)$, i.e., $\bar{K} = \{\bar{x} \in \Lambda(X): \bar{x} \geq \bar{\theta}\}$.■

Note: In the sequel the VVSS $\Lambda(X)$ ordered by (3.2.1), will be referred to an *ordered vector valued sequence space abbreviated* as OVVSS.

Let us begin with some simple properties related to the ordering of the particular VVSS $\Omega(X)$ and $\phi(X)$.

Proposition 3.2.3: Let X be a Riesz space. Then $\Omega(X)$ and $\phi(X)$ are Riesz spaces such that $\phi(X)$ is an ideal in $\Omega(X)$.

Proof: The space $\Omega(X)$ is clearly a Riesz space, for if $\bar{x}, \bar{y} \in \Omega(X)$ then $\bar{x} \vee \bar{y} = \{x_n \vee y_n\} \in \Omega(X)$; indeed, by the definition of ordering of $\Omega(X)$, $\bar{x}, \bar{y} \leq \{x_n \vee y_n\}$ in $\Omega(X)$ and if $\bar{a} \in \Omega(X)$ is such that $\bar{x}, \bar{y} \leq \bar{a}$ in $\Omega(X)$, then $\{x_n \vee y_n\} \leq \bar{a}$. Thus $\bar{x} \vee \bar{y} = \{x_n \vee y_n\}$ is in $\Omega(X)$.

For the space $\phi(X)$, consider $\bar{x}, \bar{y}$ in $\phi(X)$. Then we can find $n_o \in \mathbb{N}$ such that $x_n \vee y_n = \theta$, for all $n \geq n_o$ and so $\{x_n \vee y_n\}$ is an element of $\phi(X)$. Now proceeding as in the preceding paragraph, we get $\bar{x} \vee \bar{y} = \{x_n \vee y_n\}$ and so $\phi(X)$ is a Riesz space.

Concerning the ideal character of $\phi(X)$ in $\Omega(X)$, observe that if $|\bar{x}| \leq |\bar{y}|$, for $\bar{x} \in \Omega(X)$ and $\bar{y} \in \phi(X)$, we have $|x_n| \leq |y_n|$, $\forall n \geq 1$. As $|y_n| = \theta$, for all but finitely many indices, $\bar{x} \in \phi(X)$. Thus $\phi(X)$ is a solid subspace of $\Omega(X)$, i.e., an ideal of $\Omega(X)$.■

Proposition 3.2.4 : If X is an order complete OVS, then so is

$\Omega(X)$. In addition, if X is also a Riesz space, then $\phi(X)$ is an order complete Riesz space.

Proof : Consider a directed subset $\bar{A}$ of $\Omega(X)$, majorized by $\bar{a}$ in $\Omega(X)$. If $A_n=\{ x_n: \bar{x}= \{x_n\} \in \bar{A} \}$ for $n \in \mathbb{N}$, then A_n is a directed subset of X majorized by a_n, for each $n \in \mathbb{N}$. As X is order complete, there exists a sequence $\{b_n\}$ in X with $b_n = \sup A_n$, for each $n \geq 1$. Write $\bar{b} = \{b_n\}$. Then $\bar{b} \in \Omega(X)$ is such that $\bar{b} = \sup \bar{A}$ in $\Omega(X)$; for if $\bar{c}$ is an upper bound of $\bar{A}$, then $c_n \geq x_n$ for each n in $\mathbb{N}$ and $\bar{x} \in \bar{A}$, yields that $c_n \geq b_n$, for each $n \geq 1$. Hence $\Omega(X)$ is an order complete OVS.

For the second part, let us first consider the set $\bar{A}$ of positive elements of $\phi(X)$, majorized by $\bar{a} = \{a_n\} \in \phi(X)$. Then proceeding as above, we get $\bar{b} = \{b_n\} \in \Omega(X)$ such that $\sup A_n = b_n$, for each n in $\mathbb{N}$. As $\theta \leq b_n \leq a_n$, for each n in $\mathbb{N}$, we obtain that $\bar{b} \in \phi(X)$ as $\bar{a} \in \phi(X)$. Thus $\sup \bar{A} = \bar{b}$, is in $\phi(X)$. Now consider an arbitrary subset $\bar{A}$ of $\phi(X)$, which is majorized by $\bar{a}$ in $\phi(X)$. Then for an arbitrary fixed element $\bar{x}_o$ of $\bar{A}$, the set $\bar{B} = \{ \bar{x} \vee \bar{x}_o - \bar{x}_o :$ $\bar{x} \in \bar{A} \}$ is a set of positive elements in $\phi(X)$, majorized by $\bar{a}-\bar{x}_o$. Hence $\sup \bar{B} = \bar{b}$, say, exists in $\phi(X)$. Clearly, $\bar{b}+\bar{x}_o$ is an upper bound of $\bar{A}$ in $\phi(X)$. Also, if $\bar{u} \in \phi(X)$ is such that $\bar{x} \leq \bar{u}$, $\forall\ \bar{x} \in \bar{A}$, then $\bar{x} \vee \bar{x}_o \leq \bar{u}$, for all $\bar{x}$ in $\bar{A}$, implies that $\bar{b} + \bar{x}_o \leq \bar{u}$, i.e., $\bar{b} + \bar{x}_o = \sup \bar{A}$ in $\phi(X)$.■

Remarks : Observe that the second part of the above proposition is also a consequence of Propositions 1.2.10(i), 3.2.3 and the first part of the Proposition 3.2.4. Also, an OVVSS $\Lambda(X)$ is not necessarily order complete even if X is so, e.g., the space c of all real convergent sequences is not order complete, for the set $\{ \bar{\alpha}^n : n \in \mathbb{N} \}$, where $\alpha^n_i=0$ if $i>n$ and for $i \leq n$, $\alpha^i_n =1$ if i is odd

and $\alpha_n^i=0$ if i is even, for $n \in \mathbb{N}$, is majorized in c, but has no supremum in c. However, the ideal restriction on $\Lambda(X)$ yields the following characterization of order completeness.

Theorem 3.2.5 : Let X and $\Lambda(X)$ be Riesz spaces such that X is also order complete. Then $\Lambda(X)$ is order complete if and only if $\Lambda(X)$ is an ideal of $\Omega(X)$.

Proof: In view of Propositions 1.2.10(i), 3.2.3 and 3.2.4, it suffices to prove the necessity. Let us, therefore, assume that $\Lambda(X)$ is order complete and consider $\bar{x}$, $\bar{y}$ in $\Omega(X)$ with $|\bar{y}| \le |\bar{x}|$, where $\bar{x} \in \Lambda(X)$. For n in $\mathbb{N}$, define a sequence $\bar{z}^n = \{ z_i^n \}$ in $\Omega(X)$ as follows :

$$z_i^n = \begin{cases} |x_i|, & i \neq n \\ y_i, & i = n. \end{cases}$$

Indeed, $\bar{z}^n = |\bar{x}| - \delta_n^{|x_n|-y_n}$, for $n \ge 1$. As $\phi(X) \subset \Lambda(X)$, $\bar{z}^n \in \Lambda(X)$; $\forall\ n \ge 1$. Also, the sequence $\{\bar{z}^n : n \ge 1\}$ is bounded below by $-|\bar{x}|$ in $\Lambda(X)$ and so by the hypothesis, $\inf \{\bar{z}^n : n \ge 1\}$, say $\bar{z}$, exists in $\Lambda(X)$. But $\inf \{\bar{z}^n : n \ge 1\} = \bar{y}$ in $\Omega(X)$, as for any other lower bound $\bar{a}$ of $\{\bar{z}^n\}$ in $\Omega(X)$, we have $a_n \le y_n$, $\forall\ n \ge 1$, i.e., $\bar{a} \le \bar{y}$. Consequently, $\bar{z} \le \bar{y}$ in $\Omega(X)$. For proving $\bar{y} = \bar{z}$, fix i in $\mathbb{N}$ and consider the set $\{z_i^n : n \ge 1\}$ in X. If $a \in X$ is such that $z_i \le a \le y_i$, then $\bar{u} = \bar{z} - \delta_i^{z_i - a}$ is a lower bound of $\{\bar{z}^n\}$ in $\Lambda(X)$ such that $\bar{u} \ge \bar{z}$, which contradicts that $\bar{z}$ is the infimum of $\{\bar{z}^n : n \ge 1\}$ in $\Lambda(X)$. Hence $y_i = z_i$, for each i in $\mathbb{N}$. Thus $\bar{y} = \bar{z}$ and so $\bar{y} \in \Lambda(X)$. This proves that $\Lambda(X)$ is an ideal of $\Omega(X)$. ■

Relating the Archimedean character and order convergence of X with that of $\Lambda(X)$, we have the following results:

Proposition 3.2.6: An ordered vector space X is Archimedean if and only if $\Lambda(X)$ is so.

Proof: For proving the necessity, consider $\bar{x}$ and $\bar{y}$ in $\Lambda(X)$ such that $\lambda\bar{x} \le \bar{y}$, for all λ in $\mathbb{R}_+$. Then $\lambda x_n \le y_n$, for each n in $\mathbb{N}$ and for all λ in $\mathbb{R}_+$. As X is an Archimedean space, we infer that $x_n \le \theta$, for each $n \ge 1$, i.e., $\bar{x} \le \bar{\theta}$ in $\Lambda(X)$. Thus $\Lambda(X)$ is Archimedean.

Conversely, if $x \in X$ and $y \in K$ are such that $\lambda x \le y$, for all λ in $\mathbb{R}_+$, then $\lambda\delta_n^x \le \delta_n^y$, for all λ in $\mathbb{R}_+$ and for each arbitrarily fixed n in $\mathbb{N}$. Consequently, $\delta_n^x \le \bar{\theta}$ by the hypothesis and so $x \le \theta$ in X, i.e., X is Archimedean.■

Proposition 3.2.7: For Riesz spaces X and $\Lambda(X)$, we have

(i) $\bar{x}^{(n)} \xrightarrow{(o)} \bar{x}$ in $\Lambda(X)$, for each $\bar{x} \in \Lambda(X)$;

(ii) $x_n^\alpha \xrightarrow{(o)} x_n$ in X, for each $n \in \mathbb{N}$ if the net $\{ \bar{x}^\alpha : \alpha \in \Lambda \}$, $\bar{x}^\alpha = \{x_n^\alpha\}$, $\alpha \in \Lambda$, order converges to $\bar{x}$ in $\Lambda(X)$; and

(iii) if X and $\Lambda(X)$ are order complete Riesz spaces and $\{\bar{x}^\alpha : \alpha\in\Lambda\}$ is an order bounded net in $\Lambda(X)$ satisfying $x_n^\alpha \xrightarrow{(o)} x_n$, for each $n \in \mathbb{N}$, then $\bar{x} \in \Lambda(X)$ and $\bar{x}^\alpha \xrightarrow{(o)} \bar{x}$ in $\Lambda(X)$.

Proof: (i) For n in $\mathbb{N}$, write

$$\bar{y}^n = |\bar{x} - \bar{x}^{(n)}| = \{ \theta,\theta,\ldots\theta,|x_{n+1}|,|x_{n+2}|,\ldots\}.$$

Then, $\bar{y}^n \downarrow \bar{\theta}$ obviously and hence $\bar{x}^{(n)} \xrightarrow{(o)} \bar{x}$ in $\Lambda(X)$.

(ii) If a net $\{\bar{x}^\alpha : \alpha \in \Lambda\}$ in $\Lambda(X)$ order converges to $\bar{x}$ in $\Lambda(X)$, then we can find a net $\{ \bar{y}^\alpha : \alpha \in \Lambda \}$ in $\Lambda(X)$ such that $\bar{y}^\alpha \downarrow \bar{\theta}$ in $\Lambda(X)$ and

$$(*) \qquad |\bar{x}^\alpha - \bar{x}| \le \bar{y}^\alpha, \ \forall \ \alpha\in \Lambda.$$

Now $\bar{y}^\alpha\downarrow \bar{\theta}$ yields that $y_n^\alpha \downarrow_\alpha$ in X, for each n in $\mathbb{N}$ and if $\theta \le a \in X$ is such that $a \le y_n^\alpha$, for some arbitrarily fixed n in $\mathbb{N}$ and for each α in Λ, then $\bar{\theta} \le \delta_n^a \le \bar{y}^\alpha$ in $\Lambda(X)$ for each α in Λ. Consequently, $a = \theta$. Thus $y_n^\alpha \downarrow_\alpha \theta$ in X, for each $n \in \mathbb{N}$. Also, we have from $(*)$

$$|x_n^\alpha - x_n| \le y_n^\alpha \ , \ \forall\ \alpha \in \Lambda \text{ and } \forall\ n \in \mathbb{N},$$

and so $x_n^\alpha \xrightarrow{(o)} x_n$, for each $n \in \mathbb{N}$.

(iii) For proving this statement , let us consider an order bounded net $\{ \bar{x}^\alpha \}$ in $\Lambda(X)$ and a point $\bar{x}$ in $\Omega(X)$ such that $x_n^\alpha \xrightarrow{(o)} x_n$ in X, for each $n \in N$. Now, choose $\bar{y} \in \Lambda(X)$ such that $|\bar{x}^\alpha| \le \bar{y}$, $\forall\ \alpha \in \Lambda$, i.e., $|x_n^\alpha| \le y_n$, for each α in Λ and n in $\mathbb{N}$. Consequently, $|x_n| \le y_n$, for each $n \in \mathbb{N}$ by Proposition 1.2.7(ii). As $\Lambda(X)$, being order complete, is an ideal in $\Omega(X)$ by Theorem 3.2.5, $\bar{x} \in \Lambda(X)$.

In order to show that $\bar{x}^\alpha \xrightarrow{(o)} \bar{x}$, define nets $\{\bar{y}^\alpha\}$ and $\{\bar{z}^\alpha\}$ in $\Omega(X)$ with $\bar{y}^\alpha = \{y_n^\alpha\}$, $\bar{z}^\alpha = \{z_n^\alpha\}$, where for $\alpha \in \Lambda$ and $n \in \mathbb{N}$,

$$(*) \qquad y_n^\alpha = \sup_{\beta \ge \alpha} x_n^\beta \quad \text{and} \quad z_n^\alpha = \inf_{\beta \ge \alpha} x_n^\beta .$$

Then from the order convergence of $\{x_n^\alpha\}$ to x_n, we have

$$(**) \qquad \sup_\alpha z_n^\alpha = x_n = \inf_\alpha y_n^\alpha \ , \ \forall\ n \in \mathbb{N},$$

cf.[175], p.44. Also, from (*),

$$\bar{y}^\alpha = \sup_{\beta \ge \alpha} \bar{x}^\beta \quad \text{and} \quad \bar{z}^\alpha = \inf_{\beta \ge \alpha} \bar{x}^\beta$$

in $\Omega(X)$ and so $|\bar{y}^\alpha| \le \bar{y}$, $|\bar{z}^\alpha| \le \bar{y}$, for each $\alpha \in \Lambda$. Thus the nets $\{\bar{y}^\alpha\}$ and $\{\bar{z}^\alpha\}$ are contained in $\Lambda(X)$ as $\Lambda(X)$ is an ideal of $\Omega(X)$. Further, by (**),

$$\sup_\alpha \bar{z}^\alpha = \bar{x} = \inf_\alpha \bar{y}^\alpha, \text{ in } \Lambda(X)$$

$$\Rightarrow \qquad \bar{u}^\alpha = \bar{y}^\alpha - \bar{z}^\alpha \downarrow \bar{\theta} \text{ in } \Lambda(X) \text{ and } |\bar{x}^\alpha - \bar{x}| \le \bar{u}^\alpha, \ \forall\ \alpha \in \Lambda.$$

Hence , $\bar{x}^\alpha \xrightarrow{(o)} \bar{x}$ in $\Lambda(X)$. ■

Proposition 3.2.8: Let X be a Riesz space possessing the boundedness property. Then $\Omega(X)$ also possesses this property.

Proof: In order to show the boundedness property of $\Omega(X)$, consider a subset $\bar{B}$ of $\Omega(X)$ for which $\lambda_n \bar{x}^n \xrightarrow{(o)} \bar{\theta}$, for any sequence $\{\bar{x}^n\} \subset \bar{B}$ and $\lambda_n \downarrow 0$ in $\mathbb{R}$. For each m in $\mathbb{N}$, write B_m for

the set of m-th co-ordinates of members of $\bar{B}$, i.e., $B_m = \{ x_m : \bar{x} \in \bar{B}\}$. We now prove the order boundedness of each B_m. Therefore, for given m in $\mathbb{N}$, consider a sequence $\{y_p\} \subset B_m$. Then, for each p in $\mathbb{N}$, $y_p = y_m^p$, the m-th co-ordinate of some $\bar{y}^p \in \bar{B}$. Now by the hypothesis,

$$\lambda_p \bar{y}^p \xrightarrow{(o)} \bar{\theta} \text{ in } \Omega(X).$$

$$\Rightarrow \qquad \lambda_p y_p = \lambda_p y_p^m \xrightarrow{(o)} \theta \text{ in } X \text{ ; cf. Proposition 3.2.7(ii).}$$

Hence, B_m is order bounded, i.e., $B_m \subset [a_m, b_m]$, for some a_m, b_m in X. Write $\bar{a} = \{a_m\}$ and $\bar{b} = \{b_m\}$. Then $\bar{a}$, $\bar{b}$ are in $\Omega(X)$ satisfying $\bar{B} \subset [\bar{a}, \bar{b}]$. This completes the proof.■

Proposition 3.2.9: If X is an order separable OVS, then so is $\Lambda(X)$.

Proof: For the order separability of $\Lambda(X)$, consider a set $\bar{B}$ in $\Lambda(X)$ for which $\bar{b} = \sup \bar{B}$ exists in $\Lambda(X)$. Write B_m for the set of m-th co-ordinates of $\bar{B}$, $m \in \mathbb{N}$. We then claim that $b_m = \sup B_m$, $\forall\, m \geq 1$; for otherwise, if there exists some m in $\mathbb{N}$ such that for some $a \in X$, $x \leq a < b_m$, $\forall x \in B_m$, then $\bar{x} \leq \bar{b} - \delta_m^{b_m - a} < \bar{b}$, for each $\bar{x} \in \bar{B}$, contradicting that $\bar{b} = \sup \bar{B}$. Hence by order separability of X, for each $m \in \mathbb{N}$, we can find a countable subset $\{x_m^{m(n)} : n \in \mathbb{N}\}$ of B_m such that for each m in $\mathbb{N}$,

$$b_m = \sup \{ x_m^{m(n)} : n \in \mathbb{N} \}.$$

Now if $\bar{A} = \{ \bar{x}^{m(n)} : m \in \mathbb{N}, n \in \mathbb{N} \}$, then $\bar{A}$ is a countable subset of $\bar{B}$ such that $\bar{b} = \sup \bar{A}$ in $\Lambda(X)$. Hence the result follows.■

Relating the projection properties of bands in X and $\Lambda(X)$, we have

Proposition 3.2.10: Let X be a Riesz space such that $\Lambda(X)$ is an ideal of $\Omega(X)$. Then

(i) The set $B_m = \{x_m : \bar{x} \in \bar{B}\}$ of m-th co-ordinates of

members of a band $\bar{B}$ of $\Lambda(X)$, is a band in X, for each m in $\mathbb{N}$. Further, $\bar{B}$ is a projection band in $\Lambda(X)$ if and only if each B_m is a projection band; and

(ii) For a band B in X, $\bar{B}_m = \{\ \delta_m^x : x \in B\}$, $m \in \mathbb{N}$, are bands in $\Lambda(X)$; and these are projection bands in $\Lambda(X)$ if and only if B is a projection band in X.

Proof:(i) As $\bar{B}$ is a band in $\Lambda(X)$, the sets B_m's are clearly linear subspaces of X. Also, for given m in $\mathbb{N}$, and y in X with $|y| \le |x_m|$, for some x_m in B_m, we have $|\delta_m^y| \le |\bar{x}|$, for some $\bar{x}$ in $\bar{B}$. Thus $\delta_m^y \in \bar{B}$, i.e., $y \in B_m$, by the definition of B_m. For proving that B_m is order closed, consider a net $\{\ x_m^\alpha : \alpha \in \Lambda\}$ in B_m such that $\theta \le x_m^\alpha \uparrow_\alpha x$ in X. If $\{\bar{x}^\alpha\} \subset \bar{B}$ is such that $\bar{x}^\alpha = \{x_m^\alpha\}$, then $\delta_m^{x_m^\alpha} \le |\bar{x}^\alpha|$, $\forall\ \alpha \in \Lambda$ and so $\delta_m^{x_m^\alpha} \in \bar{B}$ for each α. Further, we claim that $\delta_m^{x_m^\alpha} \uparrow \delta_m^x$ in $\Lambda(X)$, for if $\bar{a} \in \Lambda(X)$ is such that $\delta_m^{x_m^\alpha} \le \bar{a}$, $\forall\ \alpha \in \Lambda$, then $\theta \le a_n$, for each n in $\mathbb{N}$ and $\theta \le x_m^\alpha \le a_m$, $\forall\ \alpha \in \Lambda$, and so $\theta \le x \le a_m$; hence $\bar{\theta} \le \delta_m^x \le \bar{a}$. As $\bar{B}$ is a band of $\Lambda(X)$, $\delta_m^x \in \bar{B}$. Consequently, $x \in B_m$. Thus B_m is a band of X.

For proving the second part of (i), let us assume that $\bar{B}$ is a projection band . Fix m in $\mathbb{N}$ and consider δ_m^u in $\Lambda(X)$ corresponding to a positive element u in X. Then, by Proposition 1.2.11(i), there exists $\bar{x}$ in $\Lambda(X)$ such that

$$\bar{x} = \sup\ \{\ \bar{w} \in \bar{B}:\ \bar{\theta} \le \bar{w} \le \delta_m^u\ \}$$

$$\Rightarrow \qquad x_m = \sup\ \{\ w \in B:\ \theta \le w \le u\ \} \text{ in } X.$$

Hence B_m is a projection band.

In order to establish the converse, consider a positive element $\bar{u}$ in $\Lambda(X)$. Then from our hypothesis, we can find $\{x_m\} \subset X$ such that

$$x_m = \sup\ \{\ w \in B_m:\ \theta \le w \le u_m\},\ \forall\ m \in \mathbb{N}.$$

If $\bar{x} = \{x_m\}$, then $\bar{\theta} \le \bar{x} \le \bar{u}$ and

(*) $\quad \bar{x} = \sup \{ \bar{w} \in \bar{B} : \bar{\theta} \le \bar{w} \le \bar{u}\}$

in $\Omega(X)$. As $\Lambda(X)$ is an ideal of $\Omega(X)$, $\bar{x} \in \Lambda(X)$. Thus $\bar{B}$ is such that (*) is satisfied for each $\bar{u} \ge \bar{\theta}$ in $\Lambda(X)$. Hence $\bar{B}$ is a projection band in view of Proposition 1.2.11(i).

(ii) For proving this part, consider a band B in X and $\bar{B}_m = \{ \delta_m^x : x \in B \}$ in $\Lambda(X)$, for a given m in $\mathbb{N}$. Then $\bar{B}_m$ is clearly a solid subspace of $\Lambda(X)$, as B is so in X. Also, for a net $\{\delta_m^x\alpha\}$ in $\bar{B}_m$ such that $\bar{\theta} \le \delta_m^x\alpha \uparrow_\alpha \bar{x}$ in $\Lambda(X)$, $x_j = \theta \ \forall \ j \ne m$ and $x_m = \sup_\alpha x_\alpha$, i.e., $x_m \in$ B. Consequently, $\bar{x} = \delta_m^x m \in \bar{B}_m$; i.e., $\bar{B}_m$ is a band of $\Lambda(X)$.

Now assume that B is a projection band in X and fix m in $\mathbb{N}$. Then for any arbitrary element $\bar{u} \ge \bar{\theta}$ in $\Lambda(X)$, we can find x in X such that

$$x = \sup \{ w \in B : \theta \le w \le u_m\},$$

cf. Proposition 1.2.11(i). This would imply that

$$\delta_m^x = \sup \{ \bar{w} \in \bar{B}_m : \bar{\theta} \le \bar{w} \le \bar{u} \}.$$

Hence, $\bar{B}_m$ is a projection band in $\Lambda(X)$, for each m in $\mathbb{N}$.

The converse would be true if $\bar{B}_m$ is a projection band for some m in $\mathbb{N}$, in view of necessity of (i).■

Making use of this result , we derive

Theorem 3.2.11: Let X and $\Lambda(X)$ be as in the above proposition. Then we have

(i) X has projection property if and only if $\Lambda(X)$ has;

(ii) X has principal projection property if and only if $\Lambda(X)$ has; and

(iii) X has sufficiently many projections if and only if $\Lambda(X)$ has.

Proof: (i) The necessity of (i) follows immediately from Proposition 3.2.10(i); whereas the sufficiency is a consequence of Proposition 3.2.10(ii).

(ii) For showing the necessity, consider a principal band $\bar{B}_{\bar{v}}$ generated by $\bar{v}=\{v_m\}$ in $\Lambda(X)$. Then using the hypothesis for the bands B_{v_m} generated by the elements v_m, $m \geq 1$, for any positive element $\bar{u} = \{u_m\}$ in $\Lambda(X)$, we can find a sequence $\{x_m\}$ in X such that

(+) $$x_m = \sup \{ u_m \wedge n|v_m|: n \geq 1\}, \ m \in \mathbb{N},$$

cf. Proposition 1.2.11(ii). Write $\bar{x} = \{x_m\}$. As $\bar{\theta} \leq \bar{x} \leq \bar{u}$, $\bar{x}$ is an element of $\Lambda(X)$. Also from (+)

$$\bar{x} = \sup \{ \bar{u} \wedge n|\bar{v}|: n \geq 1\}.$$

Thus $\Lambda(X)$ possesses the principal projection property.

In order to prove the converse, observe that for given m in $\mathbb{N}$, the band $\bar{B}_{\delta_m^x}$ generated by δ_m^x in $\Lambda(X)$, corresponding to a principal band B_x in X, we have the existence of $\bar{z}$ in $\Lambda(X)$ such that

(++) $$\bar{z} = \sup \{ \delta_m^y \wedge n|\delta_m^x|: n=1,2,\ldots\},$$

where y is an arbitrarily chosen positive element in X. Thus we have from (++),

$$z_m = \sup \{ y \wedge n|x|: n=1,2,\ldots\}$$

in X. Consequently, B_x is a projection band in X and (ii) follows.

Aliter: Use of Proposition 3.2.10 enables to give an alternative proof of this part which we present here. Invoking the notations of the proof of the Proposition 3.2.10 and the above proof, we observe that for given $m \in \mathbb{N}$, $B_m = B_{v_m}$, for the principal band $\bar{B}$

$= \bar{B}_{\bar{v}}$. Indeed, the inclusion $B_{v_m} \subset B_m$ is immediate from the facts that $v_m \in B_m$, B_{v_m} is the smallest band containing v_m and Proposition 3.2.10(i); and for the other inclusion, if $x_m \in B_m$, then for the corresponding $\bar{x}$ in $\bar{B}_{\bar{v}}$, we have a net $\{\bar{x}^{\alpha}\} \subset I_{\{\bar{v}\}}$ with $\bar{x}^{\alpha} \le \lambda_{\alpha}|\bar{v}|$, $\forall\ \alpha \in \Lambda$, $\{\lambda_{\alpha}\} \subset \mathbb{R}^{+}$, such that $\bar{\theta} \le \bar{x}^{\alpha} \uparrow |\bar{x}|$; consequently , $\theta \le x_m^{\alpha} \uparrow |x_m|$, and $x_m^{\alpha} \le \lambda_{\alpha} |v_m|$, $\forall\ \alpha \in \Lambda$, i.e., $x_m \in B_{v_m}$. Hence we get $B_m \equiv B_{v_m}$. Now apply Proposition 3.2.10(i) to get the necessity of (ii).

For proving the sufficiency, let us note that the principal band $\bar{B}_{\delta_m^x}$ generated by δ_m^x, $x \in X$, $m \in \mathbb{N}$, is the set $\{\delta_m^y$: $y \in B\}$, where $B \equiv B_x$. Indeed, if $\bar{B} = \{\delta_m^y$: $y \in B$ }, then $\bar{B}$ is a band containing δ_m^x by Proposition 3.2.10(ii) and so $\bar{B}_{\delta_m^x} \subset \bar{B}$. For the other inclusion, consider an element δ_m^y in $\bar{B}$ with $y \in B$. Then, we can find a net $\{x_{\alpha}\}$ in $I_{\{x\}}$ with $x_{\alpha} \le \lambda_{\alpha}|x|$, $\forall\ \alpha \in \Lambda$, $\{\lambda_{\alpha}\} \subset \mathbb{R}^{+}$ such that $\theta \le x_{\alpha} \uparrow |y|$ in X and so $\bar{\theta} \le \delta_m^{x_{\alpha}} \uparrow |\delta_m^y|$, where $\delta_m^{x_{\alpha}} \le \lambda_{\alpha}\delta_m^{|x|} = \lambda_{\alpha}|\delta_m^x|$, yields that $\{\delta_m^{x_{\alpha}}\}$ is contained in $I_{\{\delta_m^x\}}$. Consequently, $\bar{B}_{\delta_m^x} \equiv \bar{B}$. Now the result follows from Proposition 3.2.10(ii).

(iii) As a non trivial projection band in X corresponds to a non trivial projection band in $\Lambda(X)$ and vice-versa, by Proposition 3.2.10; (iii) follows immediately from the definition of sufficiently many projections.■

Remark: We may point out here that an ideal $\Lambda(X)$ becomes order complete in case X is so; and an order complete Riesz space always satisfies the projection property and so the principal projection property and has sufficiently many projections. Thus

the above results have relevance only in the case when X is not order complete. In such a situation, it is natural to look for the VVSS which are ideals in $\Omega(X)$. An example of such a space is provided in

Example 3.2.12: For an arbitrary Riesz space X, consider the space $c_o(X)$ defined in (3.4.4) as

$$c_o(X) = \{ \bar{x} = \{x_n\}: x_n \xrightarrow{(o)} \theta \text{ in } X \}.$$

Then, this space is an ideal in $\Omega(X)$ irrespective of X being order complete, cf. Proposition 3.4.5(i).

3.3 <u>Topological Properties of $\lambda(X)$</u>: Let us reproduce from Section 4 of Chapter 1, the (VVSS) $(\lambda(X), T_{\lambda(X)})$ as

(3.3.1) $\lambda(X)= \{\bar{x} =\{x_i\}: x_i \in X,\ i\geq 1 \text{ and } \{p(x_i)\} \in \lambda, \text{ for all } p \in \mathfrak{D} \}$,

where λ is a normal SVSS, (X,τ) a locally convex space with τ being generated by the family $\mathfrak{D}$ of seminorms; and $T_{\lambda(X)}$ is generated by the family $\{P_S: S \in \zeta, p \in \mathfrak{D} \}$ of seminorms with

(3.3.2) $P_S(\bar{x}) =(p_S \circ p)(\bar{x})=p_S(\{p(x_i)\})=\sup\limits_{\{\beta_i\}\in S} \sum\limits_{i=1}^{\infty} |\beta_i| p(x_i),\ \bar{x}\in\lambda(X)$.

Here ζ is a family of normal hulls of balanced, convex, $\sigma(\lambda^{x},\lambda)$-bounded subsets of λ^{x}, covering λ^{x}.

As mentioned in the introductory remarks of this chapter, we prove in this section the impact of topologies of λ and X on $T_{\lambda(X)}$ and vice-versa. Let us begin with

Proposition 3.3.3: Let (X,τ) be a locally solid Riesz space with $\mathfrak{D}$ consisting of Riesz seminorms. Then $\lambda(X)$ is an ideal in $\Omega(X)$ and $T_{\lambda(X)}$ is a locally convex solid topology on $\lambda(X)$.

Proof: For proving the solid subspace character of $\lambda(X)$ in $\Omega(X)$, let us note that $\theta \leq p(y_i) \leq p(x_i)$, $\forall\ i \in \mathbb{N}$ and for all $p \in \mathfrak{D}$, if $|\bar{y}|\leq|\bar{x}|$ in $\Omega(X)$. Let us assume now that $\bar{x} \in \lambda(X)$. Then $\{p(x_i)\}\in \lambda$ by the definition of $\lambda(X)$ and so $\{p(y_i)\}\in \lambda$ for any $\bar{y}$ in $\Omega(X)$

with $|\bar{y}| \le |\bar{x}|$, by our restriction of normality on λ , i.e., $\bar{y} \in \lambda(X)$.

In order to show that $T_{\lambda(X)}$ is a locally convex solid topology, consider $p \in \mathfrak{D}$ and $S \in \zeta$. Then by our hypothesis, $p(x_i) \le p(y_i)$, for each i in $\mathbb{N}$, and for each $\bar{x}$, $\bar{y} \in \lambda(X)$ with $|\bar{x}| \le |\bar{y}|$ and so

$$P_S(\bar{x}) = \sup_{\{\beta_i\}\in S} \sum_{i=1}^{\infty} |\beta_i| p(x_i) \le \sup_{\{\beta_i\}\in S} \sum_{i=1}^{\infty} |\beta_i| p(y_i) = P_S(\bar{y}).$$

Thus the seminorms generating the topology $T_{\lambda(X)}$ are Riesz seminorms and hence $T_{\lambda(X)}$ is a locally convex solid topology. This completes the proof. ∎

A partial converse of the above result is contained in

Proposition 3.3.4: Let X be a Riesz space equipped with a locally convex topology τ generated by the family $\mathfrak{D}$ of seminorms. Then τ is a locally convex solid topology provided $T_{\lambda(X)}$ is so.

Proof: Let us consider $p \in \mathfrak{D}$ and $x,y \in X$ with $|x| \le |y|$. Now, choose $S \in \zeta$ for which $M = \sup \{|\beta_{i_o}|: \bar{\beta} \in S\} > 0$, for a given i_o in $\mathbb{N}$. Then

$$M\, p(x) = P_S(\delta^x_{i_o}) = (p_S \circ p)(\delta^x_{i_o}) \le (p_S \circ p)(\delta^y_{i_o}) = P_S(\delta^y_{i_o}) = M\, p(y)$$

yields that p is a Riesz seminorm. As $p \in \mathfrak{D}$ is arbitrary, τ is a locally convex solid topology. ∎

From now onwards in this section, we assume that the pair (X,τ) *stands for a Hausdorff locally convex solid Riesz space with the topology τ generated by the family $\mathfrak{D}$ of Riesz seminorms and the sequence space λ is equipped with the normal topology* $\eta(\lambda,\lambda^x)$, defined corresponding to the family ζ consisting of balanced, convex and normal hulls of singleton sets in λ^x. In this case, for $\bar{\beta}$ in λ^x with S as the balanced, convex

and normal hull of $\bar{\beta}$ and $p \in \mathfrak{D}$, we write $P_S \equiv P_{\bar{\beta}}$ which takes the form

$$P_{\bar{\beta}}(\bar{x}) = \sum_{i=1}^{\infty} |\beta_i| p(x_i), \ \bar{x} \in \lambda(X).$$

Interrelating various properties of Riesz seminorms, namely, σ-Lebesgue, Lebesgue, pre-Lebesgue, σ-Fatou and Fatou; cf. Definition 1.3.11; for the families $\mathfrak{D}_\lambda$ and $\mathfrak{D}_{\lambda(X)}$, we have

Proposition 3.3.5: $p \in \mathfrak{D}$ satisfies the σ-Lebesgue property in X if and only if the corresponding seminorms $P_{\bar{\beta}}$, $\bar{\beta} \in \lambda^X$, satisfy the same.

Proof: For proving the necessity, consider $\bar{\beta}=\{\beta_i\} \in \lambda^X$ and a sequence $\{\bar{x}^n\}$ in $\lambda(X)$ such that $\bar{x}^n \downarrow \bar{\theta}$ in $\lambda(X)$. Then $x_i^n \downarrow \theta$, for each $i \in \mathbb{N}$ by Proposition 3.2.7(ii) and so $p(x_i^n) \longrightarrow 0$ as $n \longrightarrow \infty$, for each $i \geq 1$. As $\sum_{i \geq 1} |\beta_i|\ p(x_i^n) < \infty$, $\forall n \geq 1$, for an arbitrary fixed $\varepsilon > 0$ and $n=1$, choose $i_o \in \mathbb{N}$ such that

$$\sum_{i=i_o+1}^{\infty} |\beta_i| p(x_i^1) < \varepsilon$$

$$\Rightarrow \qquad 0 \leq P_{\bar{\beta}}(\bar{x}^n) = \sum_{i=1}^{\infty} |\beta_i| p(x_i^n) \leq \sum_{i=1}^{i_o} |\beta_i|\ p(x_i^n) + \varepsilon, \ \forall\ n \geq 1.$$

$$\Rightarrow \qquad P_{\bar{\beta}}(\bar{x}^n) \longrightarrow 0 \text{ as } n \longrightarrow \infty, \text{ as } p(x_i^n) \longrightarrow 0, \text{ for each } i \geq 1.$$

Thus $P_{\bar{\beta}}$ satisfies the σ-Lebesgue property.

To establish the converse, consider a sequence $\{x_n\}$ in X such that $x_n \downarrow \theta$ in X. For given i_o in $\mathbb{N}$, let us take an element $\bar{\beta}$ in λ^X for which $\beta_{i_o} \neq 0$. Then $\delta_{i_o}^{x_n} \downarrow \bar{\theta}$ in $\lambda(X)$ and so $P_{\bar{\beta}}(\delta_{i_o}^{x_n}) \longrightarrow 0$, as $n \longrightarrow \infty$. But $P_{\bar{\beta}}(\delta_{i_o}^{x_n}) = |\beta_{i_o}|\ p(x_n)$, $\forall\ n \geq 1$ and hence $p(x_n) \longrightarrow 0$, i.e., p is a σ-Lebesgue seminorm.∎

Proposition 3.3.6: $p \in \mathfrak{D}$ satisfies the Lebesgue property if and only if the corresponding seminorms $P_{\bar{\beta}}$ for $\bar{\beta} \in \lambda^X$, satisfy the same.

Proof: For proving that $P_{\bar{\beta}}$ satisfies the Lebesgue property if p does, consider a net $\{ \bar{x}^{\alpha} \}$ in $\lambda(X)$ with $\bar{x}^{\alpha} \downarrow \bar{\theta}$ in $\lambda(X)$. Then $x_i^{\alpha} \downarrow_{\alpha} \theta$ in X, for each $i \geq 1$. Now, for an arbitrary fixed $\varepsilon > 0$, as well as for fixed α_o, we can find $i_o \in \mathbb{N}$ such that

$$\sum_{i=i_o+1}^{\infty} |\beta_i|\ p(x_i^{\alpha_o}) < \varepsilon.$$

$$\Rightarrow \qquad P_{\bar{\beta}}(\bar{x}^{\alpha}) = \sum_{i=1}^{\infty} |\beta_i|\ p(x_i^{\alpha}) \leq \sum_{i=1}^{i_o} |\beta_i|\ p(x_i^{\alpha}) + \varepsilon,\ \forall\ \alpha \geq \alpha_o .$$

As $\sum_{i=1}^{i_o} |\beta_i|\ p(x_i^{\alpha}) \longrightarrow 0$, we can find $\alpha_1 \geq \alpha_o$ such that $\sum_{i=1}^{i_o} |\beta_i|\ p(x_i^{\alpha}) < \varepsilon$, $\forall\ \alpha \geq \alpha_1$. Hence

$$P_{\bar{\beta}}(\bar{x}^{\alpha}) < 2\varepsilon,\ \forall\ \alpha \geq \alpha_1,$$

i.e., $P_{\bar{\beta}}(\bar{x}^{\alpha}) \longrightarrow 0$.

The converse follows by proceeding on lines similar to the proof of the converse part of the preceding proposition by considering a net in place of the sequence.■

Proposition 3.3.7 : $p \in \mathfrak{D}$ possesses the pre-Lebesgue property if and only if $P_{\bar{\beta}}$, $\bar{\beta} \in \lambda^{x}$, have the same.

Proof: Let us assume that each $p \in \mathfrak{D}$ has the pre-Lebesgue property and consider an order bounded disjoint sequence $\{\bar{x}^n\}$ in $\lambda(X)$. Then $|\bar{x}^n| \leq \bar{x}$, for each $n \in \mathbb{N}$ and for some $\bar{x}=\{x_i\}$ in $\lambda(X)$, yields that $\{x_i^n\}$ is an order bounded disjoint sequence in X, for each $i \in \mathbb{N}$. Hence by our hypothesis, $p(x_i^n) \longrightarrow 0$, as $n \longrightarrow \infty$, for all i in $\mathbb{N}$ and for each p in $\mathfrak{D}$. Now fixing $p \in \mathfrak{D}$ and $\bar{\beta} \in \lambda^{x}$ arbitrarily, for given $\varepsilon > 0$, choose $i_o \in \mathbb{N}$ such that $\sum_{i=i_o+1}^{\infty} |\beta_i|\ p(x_i) < \varepsilon$. Then

$$P_{\beta}(\bar{x}^n) = \sum_{i=1}^{\infty} |\beta_i|\ p(x_i^n) \leq \sum_{i=1}^{i_o} |\beta_i|\ p(x_i^n) + \varepsilon$$

$$\Rightarrow \qquad P_{\beta}(\bar{x}^n) \longrightarrow 0, \text{ as } p(x_i^n) \longrightarrow 0,\ 1 \leq i \leq i_o .$$

For the other part, observe that for any disjoint order

bounded sequence $\{x_n\}$ in X, $\{\delta_i^{x_n}\}$ is an order bounded disjoint sequence in $\lambda(X)$ for each $i \geq 1$. Then for $\bar{\beta} \in \lambda^X$, with $\beta_{i_o} \neq 0$ and $p \in \mathfrak{D}$, we have

$$P_{\bar{\beta}}(\delta_{i_o}^{x_n}) \longrightarrow 0, \text{ as } n \longrightarrow \infty$$

$$\Rightarrow \qquad p(x_n) \longrightarrow 0.$$

Thus p satisfies the pre-Lebesgue property.■

Proposition 3.3.8: p is a σ-Fatou seminorm if and only if $P_{\bar{\beta}}$ for $\bar{\beta}$ in λ^X are so.

Proof: Let p satisfy the σ-Fatou property. Then for proving the σ-Fatou property of $P_{\bar{\beta}}$ for $\bar{\beta} \in \lambda^X$, let us consider a positive increasing sequence $\{\bar{x}^n\}$ in $\lambda(X)$ such that $\bar{x}^n \uparrow \bar{x}$, for some $\bar{x}$ in $\lambda(X)$. Then $\theta \leq x_i^n \uparrow x_i$ in X, $\forall\ i \geq 1$ and so $p(x_i^n) \underset{n}{\uparrow} p(x_i)$, for each $i \in \mathbb{N}$. Consequently,

$$(*) \qquad \sum_{i=1}^{m} |\beta_i|\ p(x_i^n) \uparrow \sum_{i=1}^{m} |\beta_i|\ p(x_i),\ \forall\ m \geq 1.$$

If $a \in \mathbb{R}$ is an upper bound of the sequence $\{P_{\bar{\beta}}(\bar{x}^n): n \geq 1\}$, then

$$\sum_{i=1}^{\infty} |\beta_i|\ p(x_i^n) \leq a,\ \forall\ n \geq 1$$

$$\Rightarrow \qquad \sum_{i=1}^{m} |\beta_i|\ p(x_i) \leq a,\ \forall\ m \geq 1,\ \text{by } (*).$$

Hence

$$P_{\bar{\beta}}(\bar{x}) = \sum_{i=1}^{\infty} |\beta_i|\ p(x_i) \leq a$$

$$\Rightarrow \qquad 0 \leq P_{\bar{\beta}}(\bar{x}^n) \uparrow P_{\bar{\beta}}(\bar{x}).$$

In order to prove the converse part, choose $\bar{\beta} \in \lambda^X$ with $\beta_{i_o} \neq 0$. Then for $\theta \leq x_n \uparrow x$ in X, $\delta_{i_o}^{x_n} \uparrow \delta_{i_o}^{x}$ in $\lambda(X)$, and so

$$|\beta_{i_o}|\ p(x_n) = P_{\bar{\beta}}(\delta_{i_o}^{x_n}) \underset{n}{\uparrow} P_{\bar{\beta}}(\delta_{i_o}^{x}) = |\beta_{i_o}|\ p(x)$$

$$\Rightarrow \qquad p(x_n) \uparrow p(x).$$

Thus p is a σ-Fatou seminorm and the proof is completed.■

Proposition 3.3.9: The family $\mathfrak{D}$ of Riesz seminorms generating the

topology τ on X, consists of Fatou-seminorms if and only if the corresponding family of seminorms on $\lambda(X)$ is of Fatou-seminorms.

Proof: The proof is omitted as it is analogous to that of the preceding proposition; indeed, replace the sequence by a net and σ-Fatou property by Fatou property.■

Concerning the spaces X and $\lambda(X)$, the preceding propositions immediately lead to

Theorem 3.3.10: (X,τ) is a σ-Lebesgue (resp. Lebesgue, pre-Lebesgue, σ-Fatou or Fatou) space if and only if $(\lambda(X),T_{\lambda(X)})$ is so.

Proof: We establish the result for σ-Lebesgue property as the other proofs follow analogously.

If (X,τ) is a σ-Lebesgue space, then $(\lambda(X),T_{\lambda(X)})$ is a σ-Lebesgue space by Proposition 3.3.5.

Conversely, assume that $(\lambda(X),T_{\lambda(X)})$ is a σ-Lebesgue space. Fix i_o in $\mathbb{N}$. Then, for $P\in \mathcal{D}_{\lambda(X)}$, define the seminorm q_p on X as

$$q_p(x) = P(\delta^x_{i_o}).$$

As P is a σ-Lebesgue seminorm , q_p would be so , for $x_n \downarrow \theta$ in X $\Rightarrow \delta^{x}_{i_o}n \downarrow \bar{\theta}$ in $\lambda(X)$ and so $q_p(x_n) = P(\delta^{x}_{i_o}n) \longrightarrow 0$, as $n \longrightarrow \infty$. If τ_1 is the topology generated by the family $\{q_p\colon P \in \mathcal{D}_{\lambda(X)}\}$, then (X,τ_1) is a σ-Lebesgue space such that the topology τ_1 is equivalent to τ.■

3.4 <u>Duals of Ordered Vector Valued Sequence Spaces</u>: The purpose of this section is two-fold, namely, (i) the investigations of the results concerning the Köthe and the sequential order duals of a general ordered vector valued sequence space, and (ii) finding the precise forms of the Köthe duals of some particular

vector valued sequence spaces defined analogous to the spaces ℓ^∞, ℓ^1, c, c_o, bv and bv_o. Accordingly, we divide this section into two parts – first one dealing with the problem (i) for an ideal $\Lambda(X)$ in $\Omega(X)$, defined corresponding to a Riesz space X and the second one incorporating the Köthe duals of the spaces $\ell^\infty(X)$, $\ell^1(X)$, c(X), $c_o(X)$, bv(X) and $bv_o(X)$, introduced in this subsection.

Duality between $\Lambda(X)$ and $[\Lambda(X)]^{so}$: *The basic assumption throughout this section is the consideration of those Riesz spaces X for which* $\langle X, X^{so}\rangle$ *forms a dual pair.*

Let us recall from Section 4 of Chapter 1, the generalized Köthe dual $\Lambda^x(X^{so})$ of $\Lambda(X)$. Since X^{so} is a Riesz space, $\Lambda^x(X^{so})$ is an OVS with respect to the co-ordinatewise ordering. With this ordering, $\Lambda^x(X^{so})$ is an order complete Riesz space, as shown in

Theorem 3.4.1: If X is an order complete Riesz space and $\Lambda(X)$ is an ideal in $\Omega(X)$, then $\Lambda^x(X^{so})$ is an order complete Riesz space.

Proof: In view of Propositions 1.2.10(i), 1.2.20(ii), 3.2.3, 3.2.4 and 3.2.5, it suffices to show that $\Lambda^x(X^{so})$ is an ideal in $\Omega(X^{so})$. For proving this statement, we first establish that $\Lambda^x(X^{so})$ is a Riesz space. Assume the contrary. Then we have an element $\bar{f}=\{f_n\}$ in $\Lambda^x(X^{so})$ such that $|\bar{f}|=\{|f_n|\}$ is not in $\Lambda^x(X^{so})$, and so for some positive element $\bar{x}=\{x_n\}$ in $\Lambda(X)$,

$$\sum_{n=1}^{\infty} \langle x_n, |f_n|\rangle = \infty .$$

Hence the sequence $\{ \sum_{j=1}^{n} \langle x_j, |f_j|\rangle \}$ is not a Cauchy sequence in $\mathbb{R}$ and thus there exists $\varepsilon > 0$ and an increasing sequence $\{n_k\}$ of positive integers such that

$$\sum_{j=n_k+1}^{n_{k+1}} \langle x_j, |f_j| \rangle \geq \varepsilon \ , \ \forall\, k \geq 1.$$

As

$$\langle x_j, |f_j| \rangle = \sup \{\ |\langle y, f_j \rangle| \ : \ |y| \leq x_j\ \},\ \forall\ j \in \mathbb{N},$$

we can find a sequence $\{y_j\}$ in X with $\theta \leq |y_j| \leq x_j$, $n_k+1 \leq j \leq n_{k+1}$, $k \geq 1$ such that

$$\sum_{j=n_k+1}^{n_{k+1}} |\ \langle y_j, f_j \rangle| \geq \varepsilon,\ \forall\ k \geq 1.$$

Now, define $\bar{z}=\{z_m\}$ in $\Omega(X)$ as

$$z_m = \begin{cases} y_m \ , & n_k+1 \leq m \leq n_{k+1}, k \geq 1 \\ \theta \ , & \text{otherwise} \end{cases} .$$

Clearly, $|\bar{z}| \leq \bar{x}$ and so $\bar{z} \in \Lambda(X)$. But

$$\sum_{j=1}^{\infty} |\langle z_j, f_j \rangle| = \sum_{k=1}^{\infty} \sum_{j=n_k+1}^{n_{k+1}} |\langle y_j, f_j \rangle| = \infty \ .$$

This contradicts the choice of $\bar{f}$ in $\Lambda^x(X^{so})$. Hence $|\bar{f}| \in \Lambda^x(X^{so})$. Thus $\Lambda^x(X^{so})$ is a Riesz space.

The solid subspace character of $\Lambda^x(X^{so})$ in $\Omega(X^{so})$ is immediate by the preceding paragraph and the following inequalities

$$\sum_{n=1}^{\infty} |g_n(x_n)| \leq \sum_{n=1}^{\infty} |f_n|(|x_n|) < \infty \ ,$$

valid for any $\bar{f}$ in $\Lambda^x(X^{so})$, $\bar{g}$ in $\Omega(X^{so})$ with $|\bar{g}| \leq |\bar{f}|$ and $\bar{x}$ in $\Lambda(X)$. ■

The main result of this section, exhibiting the relationship of the generalized Köthe dual of $\Lambda(X)$ with its sequential order dual, is contained in

Theorem 3.4.2: Let X be an order complete Riesz space such that $\Lambda(X)$ is an ideal in $\Omega(X)$. Then a linear functional $\bar{f}$ on $\Lambda(X)$ is in $[\Lambda(X)]^{so}$ if and only if there is a unique $\bar{F}$ in $\Lambda^x(X^{so})$,

$\bar{F}=\{f_n\}$, $f_n \in X^{so}$ for all $n \geq 1$ such that

(3.4.3) $$\bar{f}(\bar{x}) = \langle \bar{x}, \bar{F} \rangle = \sum_{n=1}^{\infty} \langle x_n, f_n \rangle \ , \ \forall\ \bar{x} = \{x_n\} \in \Lambda(X).$$

Moreover, the mapping π, defined by the relation $\pi(\bar{f})=\bar{F}$, where $\bar{f}$ and $\bar{F}$ are as in (3.4.3), is a normal Riesz isomorphism from $[\Lambda(X)]^{so}$ onto $\Lambda^{x}(X^{so})$.

Proof: In order to prove the necessity, we restrict ourselves to the positive members of $[\Lambda(X)]^{so}$ as it is a Riesz space. Let us, therefore, consider $\bar{f} \in [\Lambda(X)]^{so}$ with $\bar{f} \geq \bar{\theta}$. As $\delta_n^{\lambda x+y} = \lambda\delta_n^x + \delta_n^y$, for $n \geq 1$, $x,y \in X$ and $\lambda \in \mathbb{R}$, we define a sequence $\{f_n\}$ of positive linear functionals on X as follows:

$$f_n(x) = \langle \delta_n^x, \bar{f} \rangle, \ x \in X.$$

Since $\bar{f}$ is sequentially order continuous , each f_n is so as for a sequence $\{x_k\}$ in X such that $x_k \downarrow \theta$ in X, $\delta_n^{x}k \downarrow \bar{\theta}$ in $\Lambda(X)$, for each $n \geq 1$ and so f_n's are sequentially order continuous as $\bar{f}$ is so. Thus $\{f_n\} \subset X^{so}$.

We next prove that $\{f_n\} \in \Lambda^{x}(X^{so})$. Let $\bar{x} \in \Lambda(X)$. Then $\bar{x}^{(n)} \xrightarrow{(o)} \bar{x}$ by Proposition 3.2.7(i) and therefore, $\bar{f}(\bar{x}^{(n)}) \longrightarrow \bar{f}(\bar{x})$.

But

(*) $$\bar{f}(\bar{x}^{(n)}) = \bar{f}(\sum_{i=1}^{n} \delta_i^{x}i) = \sum_{i=1}^{n} \bar{f}(\delta_i^{x}i) = \sum_{i=1}^{n} \langle x_i, f_i \rangle \ , \ \forall\ n \in \mathbb{N}.$$

Hence

(**) $$\bar{f}(\bar{x}) = \lim_{n\to\infty} \bar{f}(\bar{x}^{(n)}) = \sum_{i=1}^{\infty} \langle x_i, f_i \rangle = \langle \bar{x}, \bar{F} \rangle,$$

where $\bar{F}=\{f_n\}$. Thus $\bar{F} \in \Lambda^{\beta}(X^{so})$. In order to complete the proof, it remains to show that $\bar{F} \in \Lambda^{x}(X^{so})$. As $|\bar{x}| \in \Lambda(X)$, we have

$$\sum_{i=1}^{\infty} |\langle x_i, f_i \rangle| \leq \sum_{i=1}^{\infty} \langle |x_i|, f_i \rangle < \infty \ , \text{ by } (**).$$

Consequently, $\bar{F} \in \Lambda^{x}(X^{so})$. Let us also note that (**) is the same as (3.4.3). Further, the uniqueness of $\bar{F}$ satisfying (3.4.3) is immediate from its construction; for if $\bar{G} = \{g_n\} \in \Lambda^{x}(X^{so})$ is such

that (3.4.3) holds for each $\bar{x}$ in $\Lambda(X)$, then

$$g_n(x) = \langle \delta_n^x, \bar{G} \rangle = \bar{f}(\delta_n^x) = \langle \delta_n^x, \bar{F} \rangle = f_n(x), \forall\, n \in \mathbb{N} \text{ and } x \in X,$$

thus $f_n = g_n$, $\forall\, n \geq 1$, i.e., $\bar{F} = \bar{G}$.

For proving the converse, consider $\bar{F} \in \Lambda^x(X^{so})$ with $\bar{F}=\{f_n\}$ and define a mapping $\bar{f}$ from $\Lambda(X)$ to $\mathbb{R}$ as follows :

$$\bar{f}(\bar{x}) = \sum_{n=1}^{\infty} \langle x_n, f_n \rangle \ , \ \bar{x} \in \Lambda(X).$$

Clearly, $\bar{f}$ is a well defined linear functional on $\Lambda(X)$. We now prove that $\bar{f}$ is sequentially order continuous. Let $\{\bar{x}^k\}$ be a sequence in $\Lambda(X)$ such that $\bar{x}^k \xrightarrow{(o)} \bar{\theta}$ in $\Lambda(X)$. Then we can find a sequence $\{\bar{y}^k\}$ in $\Lambda(X)$ with $|\bar{x}^k| \leq \bar{y}^k \downarrow \bar{\theta}$ in $\Lambda(X)$. Since $|\bar{F}|=\{|f_n|\}$ is in $\Lambda^x(X^{so})$ by Theorem 3.2.1 , $\sum_{i=1}^{\infty} \langle y_i^1, |f_i| \rangle < \infty$. Hence for an arbitrarily chosen $\varepsilon > 0$, there exists $i_o \in \mathbb{N}$ such that

$$\sum_{i=i_o+1}^{\infty} \langle y_i^1 \ , |f_i| \rangle < \varepsilon/2$$

$$\Rightarrow \qquad \sum_{i=i_o+1}^{\infty} \langle y_i^k \ , |f_i| \rangle < \varepsilon/2 \ , \ \forall\, k \geq 1.$$

As $f_n \in X^{so}$, $\forall\, n \geq 1$ and $y_i^k \downarrow \theta$,$\forall\, i \geq 1$, $\langle y_i^k, |f_i| \rangle \longrightarrow 0$ as $k \to \infty$, in $\mathbb{R}$, $\forall\, i \geq 1$. Thus we can find $k_o \in \mathbb{N}$ such that

$$\sum_{i=1}^{i_o} \langle y_i^k, |f_i| \rangle < \varepsilon/2 \ , \ \forall\, k \geq k_o.$$

Consequently,

$$\begin{aligned} |\bar{f}(\bar{x}^k)| = \left| \sum_{i=1}^{\infty} \langle x_i^k, f_i \rangle \right| &\leq \sum_{i=1}^{\infty} \langle |x_i^k|, |f_i| \rangle \\ &\leq \langle \bar{y}^k, |\bar{F}| \rangle \\ &\leq \sum_{i=1}^{i_o} \langle y_i^k, |f_i| \rangle + \varepsilon/2 \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon, \ \forall\, k \geq k_o \end{aligned}$$

$\Rightarrow$ $\qquad \bar{f}$ is sequentially order continuous.

Now $\bar{f}$ would be in $[\Lambda(X)]^{so}$ provided $\bar{f}$ is order bounded. Since X, being order complete, is Archimedean, $\Lambda(X)$ is so by Proposition

3.2.6. Further, $\mathbb{R}$ has boundedness property and hence $\bar{f} \in [\Lambda(X)]^{b}$, by Proposition 1.2.23(i).

For the last statement, observe that the mapping π from $[\Lambda(X)]^{so}$ to $\Lambda^{x}(X^{so})$, such that $\pi(\bar{f})=\bar{F}$, is well defined since each $\bar{f}$ in $[\Lambda(X)]^{so}$ corresponds to a unique $\bar{F}$. π is clearly linear, one-one and onto. Further, π and π^{-1} are both positive. Thus π is a Riesz isomorphism. It is normal as its kernel in $[\Lambda(X)]^{so}$ given by $N_{\pi} = \{\bar{f} \in [\Lambda(X)]^{so} : \pi(\bar{f}) = \bar{\theta}\} = \{\bar{\theta}\}$, is a band in $[\Lambda(X)]^{so}$; cf. Proposition 1.2.24. Hence the result is completely established.■

<u>Duals of Certain OVVSS</u>: Using the ordering of an order complete Riesz space X, we introduce in this subsection, certain OVVSS which are analogous to the well-known scalar valued sequence spaces ℓ^{1}, ℓ^{∞}, c, c_{o}, bv, bv_{o} and cs (cf. [96]). Besides finding their order structural properties with respect to the co-ordinatewise ordering here, we also investigate their generalized Köthe duals corresponding to the dual pair $\langle X, X^{so}\rangle$.

Let us assume throughout the subsection that X is an order complete Riesz space such that $\langle X, X^{so}\rangle$ forms a dual pair.

We now define

$$\ell^{1}(X) = \left\{ \{x_n\} \subset X : \{\sum_{i=1}^{n} |x_i|\} \text{ order converges in } X \right\},$$

$$\ell^{\infty}(X) = \left\{ \{x_n\} \subset X : \sup_{n\geq 1}\{|x_n|\} \text{ exists in } X \right\},$$

$$c(X) = \left\{ \{x_n\} \subset X : \{x_n\} \text{ order converges in } X \right\},$$

$$c_{o}(X) = \left\{ \{x_n\} \subset X : \{x_n\} \text{ order converges to } \theta \text{ in } X \right\}, \tag{3.4.4}$$

$$bv(X) = \left\{ \{x_n\} \subset X : \{\sum_{i=1}^{n} |x_{i+1}-x_i|\} \text{ order converges in } X \right\},$$

$$bv_{o}(X) = \left\{ \{x_n\} \subset X : \{\sum_{i=1}^{n} |x_{i+1}-x_i|\} \text{ order converges in } X \text{ and } x_n \xrightarrow{(o)} \theta \text{ in } X \right\},$$

$= bv(X) \cap c_o(X)$,

$cs(X) = \{ \{x_n\} \subset X : \{\sum_{i=1}^{n} x_i\}$ order converges in $X\}$.

Replacing X by X^{so} in (3.4.4), we have analogous OVVSS defined corresponding to X^{so}.

Concerning these spaces, we have

Proposition 3.4.5: (i) $\ell^1(X)$, $c_o(X)$ and $\ell^\infty(X)$ are ideals of $\Omega(X)$ such that $\phi(X) \subset \ell^1(X) \subset c_o(X) \subset \ell^\infty(X)$;

(ii) $c(X)$, $bv(X)$, $bv_o(X)$ are Riesz subspaces of $\Omega(X)$ containing $\phi(X)$ such that $bv_o(X) \subset bv(X) \subset c(X) \subset \ell^\infty(X)$;

(iii) $cs(X)$ is an ordered vector subspace of $\Omega(X)$ such that $\phi(X) \subset cs(X) \subset \ell^\infty(X)$.

Proof: (i) For $\bar{x} \in \phi(X)$, we have $x_n = \theta$, for $n > n_o$ for some n_o in $\mathbb{N}$, and so $\sum_{i=1}^{n} |x_i| \xrightarrow{(o)} \sum_{i=1}^{n_o} |x_i|$ as $n \longrightarrow \infty$, yields that $\phi(X) \subset \ell^1(X)$.

For proving $\ell^1(X) \subset c_o(X)$, let us consider $\bar{x} = \{x_n\}$ in $\ell^1(X)$. Then $\sum_{i=1}^{n} |x_i| \uparrow x$, for some x in X and therefore

$$|x_n| = \sum_{i=1}^{n} |x_i| - \sum_{i=1}^{n-1} |x_i| \xrightarrow{(o)} \theta$$

$\Rightarrow \quad \{x_n\} \in c_o(X)$.

As any order convergent sequence is order bounded, $c_o(X) \subset \ell^\infty(X)$. Thus $\phi(X) \subset \ell^1(X) \subset c_o(X) \subset \ell^\infty(X)$.

Whereas order completeness of X gives the linear subspace character of $\ell^1(X)$ and $\ell^\infty(X)$; in view of Proposition 1.2.7(i), $c_o(X)$ is a linear subspace of $\Omega(X)$.

For proving the ideal character of these spaces, consider $\bar{x}$, $\bar{y}$ in $\Omega(X)$ with $|\bar{y}| \le |\bar{x}|$.

If $\bar{x} \in \ell^1(X)$, then $\sum_{i=1}^{n} |x_i| \uparrow x$, for some $x \in X$. But $\sum_{i=1}^{n} |y_i| \le \sum_{i=1}^{n} |x_i| \le x$, $\forall n \ge 1$, implies that $\{\sum_{i=1}^{n} |y_i|\}$ is an order

bounded sequence in the order complete space X and hence $\sum_{i=1}^{n} |y_i| \uparrow y$, for some $y \in X$, cf. Proposition 1.2.7(iii). Thus $\{y_i\} \in \ell^1(X)$.

If $\bar{x} \in c_o(X)$, then $|y_n| \le |x_n| \xrightarrow{(o)} \theta \Rightarrow y_n \xrightarrow{(o)} \theta$ in X, i.e., $\bar{y} \in c_o(X)$.

If $\bar{x} \in \ell^\infty(X)$, then clearly $\{y_n\}$ is an order bounded sequence in X and so its supremum exists in X as X is order complete. Thus $\bar{y} \in \ell^\infty(X)$.

(ii) By the definition of c(X) and Proposition 1.2.7(i), it is a Riesz subspace of $\Omega(X)$ containing $c_o(X)$.

Let us now consider the space bv(X) which is a linear subspace of $\Omega(X)$, for if $\bar{x}, \bar{y} \in bv(X)$, then the inequalities

$$\sum_{i=1}^{n} |x_{i+1}+y_{i+1}-x_i-y_i| \le \sum_{i=1}^{n} |x_{i+1}-x_i| + \sum_{i=1}^{n} |y_{i+1}-y_i|, \ \forall\ n \ge 1,$$

along with the order completeness of X, yield that $\bar{x}+\bar{y} \in bv(X)$. Similarly, $\alpha\bar{x}$ is in bv(X) for $\alpha \in \mathbb{R}$ and $\bar{x} \in bv(X)$. Riesz subspace character of the space follows from the inequalities

$$\sum_{i=1}^{n} ||x_{i+1}|-|x_i|| \le \sum_{i=1}^{n} |x_{i+1}-x_i|, \ \forall\ n \in \mathbb{N},\ \bar{x} \in bv(X),$$

and the order completeness of X. Also, $\phi(X) \subset bv(X)$, for if $\bar{x} \in \phi(X)$, then

$$\sum_{i=1}^{n} |x_{i+1}-x_i| \xrightarrow{(o)} \sum_{i=1}^{n_o} |x_{i+1}-x_i| \text{ in X, where } x_n = \theta,\ \forall\ n > n_o.$$

As $bv_o(X) = bv(X) \cap c_o(X)$, the proof for $bv_o(X)$ follows analogously.

Further, let $\bar{x} \in bv(X)$ and $z_n = \sum_{j=1}^{n} |x_{j+1}-x_j|$, $n \ge 1$. Then there exists a sequence $\{y_n\}$ in X and an element x in X such that $|z_n - x| \le y_n$, for each n in $\mathbb{N}$ with $y_n \downarrow \theta$ in X. Hence

$$|z_n - z_m| \le |z_n - x| + |z_m - x| \le y_n + y_m \le y_k\ ,\ \forall\ n,m \ge k$$

$\Rightarrow$ $\{z_n\}$ is an order Cauchy sequence in X

$\Rightarrow$ $\{x_n\}$ is an order Cauchy sequence in X, as

$$|x_{n+p}-x_n| \le \sum_{j=1}^{p} |x_{n+j}-x_{n+j-1}| = \sum_{j=1}^{n+p-1} |x_{j+1}-x_j| - \sum_{j=1}^{n-1} |x_{j+1}-x_j|,$$

for $n,p \in \mathbb{N}$. Consequently, $\{x_n\} \in c(X)$ by Proposition 1.2.13(iii). Thus $bv(X) \subset c(X)$.

The inclusion $c(X) \subset \ell^{\infty}(X)$ follows from the fact that every order convergent sequence is order bounded.

(iii) This part is obvious by the definition of ordering under consideration, namely, co-ordinatewise ordering and Proposition 1.2.7(i).■

Remarks: Note that if $X = \mathbb{R}$, the spaces $c(X)$, $bv(X)$ and $bv_o(X)$ are not ideals of $\Omega(X)$, and the space $cs(X)$ is not even a Riesz subspace of $\Omega(X)$. Indeed, if we take $\bar{x}=\{1,1,1,....\}$ and $\bar{y}=\{1,0,1,0,...\}$, then $\bar{x} \in c(X)$ and $bv(X)$ with $|\bar{y}| \le |\bar{x}|$, but $\bar{y}$ is neither in $c(X)$, nor in $bv(X)$; also for $bv_o(X)$, $\bar{u}=\{1,1/2,1/3,....\} \in bv_o(X)$ and $\bar{v}=\{1,-1/2,1/3,-1/4...\} \notin bv_o(X)$, though $|\bar{v}|=|\bar{u}|$. For the space $cs(X)$, consider the sequences $\{(-1)^n/n\}$ and $\{1/n\}$, where $\{(-1)^n/n\}$ is an element of $cs(X)$ but $\{1/n\}$ is not in the space. Thus $cs(X)$ is not a Riesz subspace of $\Omega(X)$.

We now find the Köthe duals of the spaces introduced in (3.4.4). Let us begin with

Proposition 3.4.6: For an order complete Riesz space X,

(i) $(\ell^{\infty}(X))^{x} = \ell^{1}(X^{so})$; (ii) $(c_o(X))^{x} = \ell^{1}(X^{so})$;

(iii) $(c(X))^{x} = \ell^{1}(X^{so})$; and (iv) $(\ell^{1}(X))^{x} = \ell^{\infty}(X^{so})$, provided every $\sigma(X^{so},X)$-bounded set in X^{so} is order bounded in X^{so}.

Proof:(i) For proving $\ell^{1}(X^{so}) \subset (\ell^{\infty}(X))^{x}$, consider $\bar{f} \in \ell^{1}(X^{so})$

and $\bar{x} \in \ell^{\infty}(X)$. Then for $k \geq 1$,

$$\sum_{n=1}^{k} |f_n(x_n)| \leq \sum_{n=1}^{k} |f_n|(|x_n|) \leq (\sum_{n=1}^{k} |f_n|)(x) \leq g(x),$$

where x in X is such that $|x_n| \leq x$, $\forall\ n \geq 1$ and g is the order limit of the sequence $\{\sum_{n=1}^{k}|f_n|\}$ in X^{so}, i.e., $\sum_{n=1}^{k} |f_n| \uparrow_k g$ in X^{so}. Thus, $\sum_{n=1}^{\infty} |f_n(x_n)| < \infty$, $\forall\ \bar{x} \in \ell^{\infty}(X)$ and so $\bar{f} \in (\ell^{\infty}(X))^{x}$.

To prove the other inclusion, let $\bar{f} \in (\ell^{\infty}(X))^{x}$ and write $g_n = \sum_{i=1}^{n}|f_i|$, for $n \geq 1$. We now show that $\{g_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$ for each x in X. Assume the contrary. Then for some x in K, there exists $\varepsilon > 0$ and an increasing sequence $\{n_k\}$ of positive integers such that

$$\varepsilon \leq g_{n_{k+1}}(x) - g_{n_k}(x) = \sum_{i=n_k+1}^{n_{k+1}} |f_i|(x) = \sum_{i=n_k+1}^{n_{k+1}} \sup\{|f_i(y)| : |y| \leq x\}, \ \forall k \geq 1.$$

Choose $y_i^k \in X$ with $|y_i^k| \leq x$ for $n_k+1 \leq i \leq n_{k+1}$, $k \geq 1$ such that

$$(*) \qquad \varepsilon \leq \sum_{i=n_k+1}^{n_{k+1}} |f_i(y_i^k)|, \ \forall\ k \geq 1.$$

Now define $\bar{z} = \{z_m\}$ in $\Omega(X)$ as follows :

$$z_m = \begin{cases} y_m^k, & n_k+1 \leq m \leq n_{k+1},\ k \geq 1 \\ \theta, & \text{otherwise} \end{cases} .$$

Then $\bar{z} \in \ell^{\infty}(X)$ as $|z_m| \leq x$, $\forall\ m \in \mathbb{N}$. Also, by (*)

$$\sum_{i=1}^{\infty} |f_i(z_i)| = \sum_{k=1}^{\infty} \sum_{m=n_k+1}^{n_{k+1}} |f_m(y_m^k)| = \infty.$$

This contradicts the choice of $\bar{f}$ in $(\ell^{\infty}(X))^{x}$. Hence, $\{g_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$ for each x in K and so for each x in X. Let us write

$$g(x) = \lim_{n \to \infty} g_n(x), \ x \in X.$$

Then g is a linear functional on X such that $g_n \uparrow g$ in X^b by Theorem 1.2.22(ii). As $\{g_n\} \subset X^{so}$, $g \in X^{so}$ by Theorem 1.2.20(ii). Hence $\bar{f} \in \ell^1(X^{so})$ and we conclude that $(\ell^{\infty}(X))^{x} \subset \ell^1(X^{so})$.

(ii) As $c_o(X) \subset \ell^\infty(X)$, in view of (i) it suffices to prove that $(c_o(X))^x \subset \ell^1(X^{so})$. Therefore, consider $\bar{f}=\{f_n\}$ in $(c_o(X))^x$ and write $g_n=\sum_{i=1}^{n}|f_i|$, $n \geq 1$. We now show that $\{g_n(x)\}$ is a bounded sequence in $\mathbb{R}$ for all $x \in K$. Let us assume the contrary. Then there exists $x \in K$ and an increasing sequence $\{n_k\}$ of positive integers such that

$$1 \leq g_{n_1}(x);$$

and

$$2^k \leq g_{n_{k+1}}(x)-g_{n_k}(x) = \sum_{i=n_k+1}^{n_{k+1}} \sup\,\{|f_i(y)|:\ |y| \leq x\},\ k \geq 1.$$

Hence we can find $y_i^k \in X$ such that $|y_i^k| \leq x$, $n_k+1 \leq i \leq n_{k+1}$, $k \geq 1$ and

$$(**)\qquad 2^k \leq \sum_{i=n_k+1}^{n_{k+1}} |\ f_i(y_i^k)|,\ \forall\ k\geq 1.$$

Now, define $\bar{z}=\{z_i\}$ and $\bar{x}=\{x_i\}$ in $\Omega(X)$ as

$$z_i=\begin{cases} y_i^k/2^k, & n_k+1 \leq i \leq n_{k+1}, k\geq 1 \\ \theta & , \text{otherwise} \end{cases} \quad ;\quad x_i=\begin{cases} x/2^k & , n_k+1\leq i \leq n_{k+1}, k\geq 1 \\ \theta & , \text{otherwise} \end{cases}.$$

Then $\bar{x} \in c(X)$ by Proposition 1.2.7(iv) and $|\bar{z}| \leq \bar{x}$. Thus $\bar{z} \in c(X)$ and also from (**)

$$\sum_{i=1}^{\infty}|f_i(z_i)| = \sum_{k=1}^{\infty} \sum_{i=n_k+1}^{n_{k+1}} |f_i(y_i^k/(2^k))| = \infty .$$

This contradicts that $\{f_i\} \in (c_o(X))^x$. Thus $(c_o(X))^x = \ell^1(X^{so})$.

(iii) Immediate from (i),(ii) and the inclusions $c_o(X) \subset c(X) \subset \ell^\infty(X)$.

(iv) For proving $\ell^\infty(X^{so}) \subset (\ell^1(X))^x$, consider $\bar{f}=\{f_n\}$ in $\ell^\infty(X^{so})$ and $\bar{x}=\{x_n\}$ in $\ell^1(X)$. Then for each $k\geq 1$, we have

$$\sum_{n=1}^{k}|f_n(x_n)| \leq \sum_{n=1}^{k}|f_n|(|x_n|) \leq g(\sum_{n=1}^{k}|x_n|) \leq g(y),\ \forall\ k \geq 1,$$

where $g= \sup_n|f_n|$ in X^{so} and $y \in X$ is such that $\sum_{n=1}^{k}|x_n| \uparrow_k y$ in X.

Hence, $\bar{f} \in (\ell^1(X))^x$.

Conversely, let $\bar{f}=\{f_n\} \in (\ell^1(X))^x$. For proving that the set $\{|f_n|: n\geq 1\}$ is order bounded in X^{so}, we show that it is $\sigma(X^{so},X)$- bounded. Assume that it is not true. Then there exist x in K and an increasing sequence $\{n_k\}$ of positive integers such that

$$k2^k \leq |f_{n_k}|(x), \ \forall\ k\geq 1$$

$$\Rightarrow \qquad k \leq |f_{n_k}|(x/2^k) = \sup\{|f_{n_k}(y)|: |y|\leq x/2^k\}, \ \forall\ k\geq 1.$$

Choose y_{n_k} in X with $|y_{n_k}| \leq x/2^k$, $\forall\ k\geq 1$ such that

$$k \leq |f_{n_k}(y_{n_k})|, \ \forall\ k\geq 1 .$$

Now define a sequence $\{z_m\}$ in X by

$$z_m = \begin{cases} y_{n_k} & , \ m = n_k \\ \theta & \text{otherwise.} \end{cases}$$

As $\sum_{k=1}^{n} x/2^k \uparrow x$, $\{\sum_{i=1}^{n} |z_i|\}$ is order bounded in X which is order complete, and hence $\bar{z}=\{z_m\} \in \ell^1(X)$; but

$$\sum_{m=1}^{\infty} |f_m(z_m)| = \sum_{k=1}^{\infty} |f_{n_k}(y_{n_k})| = \infty.$$

This contradicts the choice of $\bar{f}$ in $(\ell^1(X))^x$. Hence the set $\{|f_n|: n\geq 1\}$ is an order bounded set in X^{so}, i.e., $\bar{f} \in \ell^\infty(X^{so})$. Consequently, $(\ell^1(X))^x \subset \ell^\infty(X^{so})$. This completes the proof.■

For the Köthe duals of OVVSS defined by the elements of the sequential order dual X^{so} of a Riesz space X, we have

Proposition 3.4.7: Let X be an order complete Riesz space such that every $\sigma(X,X^{so})$- bounded set is order bounded in X. Then

(i) $(\ell^1(X^{so}))^x = \ell^\infty(X)$; (ii) $(\ell^\infty(X^{so}))^x = \ell^1(X)$;

(iii) $(c_o(X^{so}))^x = \ell^1(X)$; and (iv) $(c(X^{so}))^x = \ell^1(X)$.

Proof: (i) The inclusion $\ell^\infty(X) \subset (\ell^1(X^{so}))^x$ follows from the inequalities

$$\sum_{n=1}^{k}|f_n(x_n)| \le \sum_{n=1}^{k}|f_n|(|x_n|) \le (\sum_{n=1}^{k}|f_n|)(x) \le g(x),\ \forall\ k\ge1,$$

valid for each $\bar{f}=\{f_n\}$ in $\ell^1(X^{so})$ and each $\bar{x}=\{x_n\}$ in $\ell^\infty(X)$, where $\sum_{n=1}^{k}|f_n| \uparrow_k g$ in X^{so} and $x \in X$ is such that $|x_n| \le x$, $\forall$ $n\ge1$.

For proving the other inclusion, namely, $(\ell^1(X^{so}))^x \subset \ell^\infty(X)$, consider $\bar{x}=\{x_n\} \in (\ell^1(X^{so}))^x$. We now establish the boundedness of the set $\{f(|x_n|):n\ge1\}$ in $\mathbb{R}$ for each positive f in X^{so}. Let us assume that the set is not so for some positive f in X^{so}. Then we can find an increasing sequence $\{n_k\}$ of positive integers such that

$$k2^k \le f(|x_{n_k}|),\ \forall\ k\ge1.$$

$$\Rightarrow \quad k \le (f/2^k)(|x_{n_k}|) = \sup\{|g(x_{n_k})|:\ |g| \le f/2^k\},\ \forall\ k\ge1.$$

$$\Rightarrow \quad k \le |g_{n_k}(x_{n_k})|,\ \forall\ k\ge1,$$

for some sequence $\{g_{n_k}\}\subset X^{so}$ with $|g_{n_k}|\le f/2^k$, for each $k\ge1$. Now proceed as in the proof of Proposition 3.4.6(iv) in order to get the result; indeed, replace x by f, f_{n_k} by x_{n_k}, y_{n_k} by g_{n_k} and define a sequence $\{h_m\}$ in X^{so} in place of the sequence $\{z_m\}$ in X analogously.

(ii) For showing the inclusion $\ell^1(X) \subset (\ell^\infty(X^{so}))^x$, consider $\bar{x}=\{x_n\}$ in $\ell^1(X)$ and $\bar{f}=\{f_n\}$ in $\ell^\infty(X^{so})$. Then we can find an element y in X and g in X^{so} such that $\sum_{n=1}^{k}|x_n| \uparrow y$ in X and $|f_n|\le g$, $\forall$ $n \ge 1$, in X^{so}. Consequently,

$$\sum_{n=1}^{k}|f_n(x_n)| \le \sum_{n=1}^{k}|f_n|(|x_n|) \le g(\sum_{n=1}^{k}|x_n|) \le g(y),\ \forall\ k\ge1$$

$$\Rightarrow \quad \sum_{n=1}^{\infty}|f_n(x_n)| < \infty.$$

Hence $\bar{x} \in (\ell^\infty(X^{so}))^x$.

Next, we prove that $(\ell^\infty(X^{so}))^x\subset \ell^1(X)$. So, let $\bar{x}=\{x_n\}$ be an element of $(\ell^\infty(X^{so}))^x$ and assume that the set

$\{f(\sum_{i=1}^{n}|x_i|):n\geq 1\}$ is not bounded in $\mathbb{R}$ for some $f\geq\theta$ in X^{so}. Hence the sequence $\{f(\sum_{i=1}^{n}|x_i|):n\geq 1\}$ can not be a Cauchy sequence in $\mathbb{R}$ and therefore, for some $\varepsilon>0$, we can find an increasing sequence $\{n_k\}$ of positive integers satisfying

$$\varepsilon \leq \sum_{i=n_k+1}^{n_{k+1}} f(|x_i|) = \sum_{i=n_k+1}^{n_{k+1}} \sup\{\, |g(x_i)|: |g|\leq f\},\ \forall\ k\geq 1.$$

Consequently, there exist $g_i^k \in X^{so}$ with $|g_i^k| \leq f$, for $n_k+1 \leq i \leq n_{k+1}$, $k\geq 1$, such that

$$\varepsilon \leq \sum_{i=n_k+1}^{n_{k+1}} |\, g_i^k(x_i)| \ ,\ k\geq 1.$$

As the set $\{g_i^k:\ n_k+1 \leq i \leq n_{k+1}, k\geq 1\}$ is order bounded in X^{so}, we define a sequence $\{h_m\}$ in $\ell^\infty(X^{so})$ as

$$h_m=\begin{cases} g_m^k, n_k+1\leq m \leq n_{k+1}, k\geq 1, \\ \theta\ ,\text{otherwise} \end{cases}.$$

However, $\sum_{k=1}^{\infty}\sum_{i=n_k+1}^{n_{k+1}} |\, g_i^k(x_i)| = \infty$. This yields a contradiction to our choice of $\bar{x}$. Hence the set $\{\sum_{i=1}^{n}|x_i|:\ n\geq 1\}$ is $\sigma(X,X^{so})$-bounded and so order bounded by our hypothesis. As X is order complete, $\sum_{i=1}^{n}|x_i| \uparrow_n x$, for some x in X, i.e., $\bar{x}\in\ell^1(X)$.

(iii) As $c_o(X^{so})\subset \ell^\infty(X^{so})$ by Proposition 3.4.5(i), we have $\ell^1(X) = (\ell^\infty(X^{so}))^X \subset (c_o(X^{so}))^X$.

In order to establish $(c_o(X^{so}))^X \subset \ell^1(X)$, consider an element $\bar{x}=\{x_n\}$ of $(c_o(X^{so}))^X$. Then for proving $\bar{x}\in\ell^1(X)$, it suffices to prove the $\sigma(X,X^{so})$-boundedness of the set $\{y_n:n\geq 1\}$, where $y_n = \sum_{i=1}^{n}|x_i|$, for each $n\in\mathbb{N}$. Assuming the contrary, we get a positive $f\in X^{so}$ and an increasing sequence $\{n_k\}$ of positive integers such that

$1 \leq f(y_{n_1});$

and

$$2^k \le f(y_{n_{k+1}}) - f(y_{n_k}) = \sum_{i=n_k+1}^{n_{k+1}} \sup\{|g(x_i)| : |g| \le f\},\ k \ge 1.$$

Now choose $g_i^k \in X^{so}$ with $|g_i^k| \le f$, $n_k+1 \le i \le n_{k+1}$, $k\ge1$ such that

$$2^k \le \sum_{i=n_k+1}^{n_{k+1}} |g_i^k(x_i)|\ ,\ \forall\ k\ge1,$$

and proceed as in the proof of Proposition 3.4.6(ii) with y_i^k being replaced by g_i^k, f_i by x_i and X by X^{so}.

(iv) Follows from (ii),(iii) and the inclusions $c_o(X^{so}) \subset c(X^{so}) \subset \ell^\infty(X^{so})$. This establishes the result completely.■

Recalling the definition of a perfect VVSS, we conclude from the above propositions, the following

Theorem 3.4.8: Let X be an order complete Riesz space for which each $\sigma(X,X^{so})$- bounded set is order bounded. Then

(i) $\ell^\infty(X)$ is a perfect VVSS; and

(ii) $\ell^1(X)$ is a perfect VVSS, provided $\sigma(X^{so},X)$-bounded sets in X^{so} are order bounded in X^{so}.

Proof:(i) Since$(\ell^\infty(X))^x = \ell^1(X^{so})$, by Proposition 3.4.6(i) and $(\ell^1(X^{so}))^x = \ell^\infty(X)$ by Proposition 3.4.7(i), the space $\ell^\infty(X)$ is a perfect space.

(ii) As $(\ell^1(X))^x = \ell^\infty(X^{so})$ by Proposition 3.4.6(iv) and $(\ell^\infty(X^{so}))^x = \ell^1(X)$ from Proposition 3.4.7(ii), $\ell^1(X)$ is a perfect VVSS.■

We now apply the above results for investigating the sequential order duals of the OVVSS $c_o(X)$, $\ell^1(X)$ and $\ell^\infty(X)$ in the form of

Theorem 3.4.9: For an order complete Riesz space X, we have

(i) $(c_o(X))^{so} \equiv \ell^1(X^{so})$; (ii) $(\ell^\infty(X))^{so} \equiv \ell^1(X^{so})$; and

(iii) $(\ell^1(X))^{so} \equiv \ell^\infty(X^{so})$, if every $\sigma(X^{so},X)$-bounded set is order bounded in X^{so}.

Proof: (i) From Theorem 3.4.2 and Proposition 3.4.5(i), $(c_o(X))^{so} \equiv (c_o(X))^{x}$, but Proposition 3.4.6(ii) says that $(c_o(X))^{x} \equiv \ell^1(X^{so})$; thus (i) is established.

(ii) $\ell^\infty(X)$ is an ideal of $\Omega(X)$ by Proposition 3.4.5(i) and so $(\ell^\infty(X))^{x} = (\ell^\infty(X))^{so}$, by Theorem 3.4.2. Hence the result follows by Proposition 3.4.6(i).

(iii) Once again by making use of Theorem 3.4.2 and Proposition 3.4.5(i), we conclude that $(\ell^1(X))^{so} \equiv (\ell^1(X))^{x}$; and so $(\ell^1(X))^{so} \equiv \ell^\infty(X^{so})$ by our hypothesis and Proposition 3.4.6(iv).■

From Theorem 3.4.2, we know that the generalized Köthe dual $\Lambda^{x}(X^{so})$ of an OVVSS $\Lambda(X)$ is an order complete Riesz subspace of $\Omega(X^{so})$ if $\Lambda(X)$ is an ideal of $\Omega(X)$. As $bv_o(X)$ is not an ideal of $\Omega(X)$, we may not apply this result to get the Riesz subspace character of $(bv_o(X))^{x}$ in $\Omega(X^{so})$. However, in particular case when $X = \mathbb{R}$, $bv_o^{\,x} = \ell^1$ which is an ideal of w, cf.[96]. In general, we have

Theorem 3.4.10: $\ell^1(X^{so})$ is the largest Riesz subspace of $(bv_o(X))^{x}$.

Proof: As $bv_o(X) \subset c_o(X)$, $\ell^1(X^{so}) \subset (bv_o(X))^{x}$, by Proposition 3.4.6(ii). Since $\ell^1(X^{so})$ is an ideal in $\Omega(X^{so})$, by Proposition 3.4.5(i), it is an ordered vector subspace of $(bv_o(X))^{x}$. For showing that it is the largest Riesz subspace of $(bv_o(X))^{x}$, consider another Riesz subspace $\Lambda(X^{so})$ of $(bv_o(X))^{x}$. Now for an element $\bar{a}=\{a_n\}$ of bv_o, with $\sum_{i=1}^{n} |a_{i+1}-a_i| \uparrow a$ in $\mathbb{R}$, and $x \in K$, if $\bar{x} = \{x_n\}$ is such that $x_n = a_n x$, $n \geq 1$, then

$$\sum_{i=1}^{n} |x_{i+1}-x_i| = (\sum_{i=1}^{n} |a_{i+1}-a_i|)(x) \uparrow ax; \text{ and } a_n x \xrightarrow{(o)} \theta \text{ in } X.$$

Hence $\bar{x} \in bv_o(X)$, for such a choice of $\{a_n\}$ in bv_o and x in X. Let $\bar{f}=\{f_n\} \in \Lambda(X^{so}) \subset (bv_o(X))^x$, then for $\{a_n\} \in bv_o$ and $x \in X$,

$$\sum_{n=1}^{\infty} |\langle |f_n|(x), a_n\rangle| = \sum_{n=1}^{\infty} ||f_n|(x_n)| < \infty$$

$$\Rightarrow \quad \{|f_n|(x)\} \in bv_o^x = \ell^1,$$

cf.[96],p.69 . Set

$$g_k = \sum_{n=1}^{k} |f_n| \quad , \quad k \geq 1.$$

Then $\{g_k(x)\}$ is a bounded increasing sequence in $\mathbb{R}$ and so we can define a linear functional g on X by

$$g(x) = \lim_{k \to \infty} g_k(x) \quad , \quad x \in X.$$

Then by Theorems 1.2.20(ii) and 1.2.22(ii), $g_k \uparrow g$ in X^{so}. Hence $\bar{f} \in \ell^1(X^{so})$ and so $\Lambda(X^{so}) \subset \ell^1(X^{so})$. This completes the proof.■

The above reslt immediately leads to

Proposition 3.4.11: Let X be an order complete Riesz space . If $(bv_o(X))^x$ is a Riesz subspace of $\Omega(X^{so})$, then

$$\text{(i)} (bv_o(X))^x = \ell^1(X^{so}); \text{ and (ii) } (bv(X))^x = \ell^1(X^{so}).$$

Proof: (i) As $\ell^1(X^{so})$ is the largest Riesz subspace of $(bv_o(X))^x$ by Theorem 3.4.10, (i) follows by our restriction on $(bv_o(X))^x$.

(ii) As $bv_o(X) \subset bv(X)$, $(bv(X))^x \subset (bv_o(X))^x = \ell^1(X^{so})$, by (i). But $bv(X) \subset \ell^\infty(X)$, by Proposition 3.4.5(ii); and so by Proposition 3.4.6(i), $\ell^1(X^{so}) = (\ell^\infty(X))^x \subset (bv(X))^x$. Hence (ii) holds.■

CHAPTER 4

o-MATRIX TRANSFORMATIONS

4.1 An Overview : This chapter exclusively involves the study of o-matrix transformations introduced on OVVSS which have underlying spaces as the Riesz spaces. In the next section, after introducing the concept of o-matrix transformation, we incorporate results dealing with the characterizations of the matrix representation of linear maps as well as their adjoints in terms of their sequential order continuity, besides deriving the necessary conditions for a matrix of linear operators to transform $\ell^1(X)$ into simple OVVSS $\Lambda(X)$. In Section 4.3, we characterize o-matrix transformations on particular OVVSS $c(X)$, $\ell^\infty(X)$, $cs(X)$ introduced in the preceding chapter. The last section, namely 4.4, includes a characterization of o-precompactness of o-matrix transformations in terms of component operators, when they are defined from an OVVSS to a topological OVVSS.

4.2 o-Matrix Transformations and Sequential Order Continuity:

Throughout this chapter , we denote by X and Y two Riesz spaces and by $\Lambda(X)$ and $\mu(Y)$ two OVVSS defined over X and Y respectively. Let us now introduce

Definition 4.2.1: A linear map Z from $\Lambda(X)$ to $\mu(Y)$ is said to be an *o-matrix transformation* if there exists a matrix $[Z_{ij}]$ of linear maps from X to Y such that for every $\bar{x}=\{x_i\}$ in $\Lambda(X)$ and for each i in $\mathbb{N}$, the sequence $\{\sum_{j=1}^{k} Z_{ij}(x_j):k \geq 1\}$ order converges to some y_i in Y such that $Z(\bar{x}) = \bar{y}$, where $\bar{y} = \{y_i\}$; and in such a case we write $Z \equiv [Z_{ij}]$.

Let us denote by $\mathfrak{L}_m(\Lambda(X),\mu(Y))$, the vector space containing all o-matrix transformations from $\Lambda(X)$ to $\mu(Y)$. The *transpose* $Z^{\perp}$ of a matrix $Z \equiv [Z_{ij}]$ of linear maps $Z_{ij}:X\longrightarrow Y$, $i,j\geq 1$, is defined as the transpose of the matrix of the adjoint maps, i.e., $Z^{\perp}\equiv [Z^*_{ji}]$.

Unless otherwise specified, *we assume in this section that* X *and* Y *are order complete Riesz spaces for which* $\langle X,X^{so}\rangle$, $\langle Y,Y^{so}\rangle$ *form dual pairs; and* $\Lambda(X)$, $\mu(Y)$ *are ideals in* $\Omega(X)$, $\Omega(Y)$ respectively. Let us now begin with the simple facts concerning o-matrix transformations contained in

Proposition 4.2.2: For the o-matrix transformation $Z \equiv [Z_{ij}]$ defined from $\Lambda(X)$ to $\mu(Y)$, the following statements hold:

(i) For $x \in X$ and $i,j \in \mathbb{N}$; $Z_{ij}(x) = (Z(\delta^x_j))_i$, the i-th co-ordinate of $Z(\delta^x_j)$;

(ii) Z is positive if and only if the Z_{ij}'s are positive for all $i,j \in \mathbb{N}$;

(iii) Z_{ij}'s are order bounded linear maps if Z is so;

(iv) Z_{ij}'s are sequentially order continuous if Z is so;

(v) if $Z=Z_1-Z_2$ where $Z_1\equiv [Z^1_{ij}]$ and $Z_2\equiv [Z^2_{ij}]$, then $Z_{ij}=Z^1_{ij}-Z^2_{ij}$; and

(vi) If $Z_1\equiv [Z^1_{ij}]$, $Z_2\equiv [Z^2_{ij}]$ are two o-matrix transformations from $\Lambda(X)$ to $\mu(Y)$, then $Z^1_{ij} \wedge Z^2_{ij} = \theta$, for each $i,j \geq 1$, if $Z_1\wedge Z_2 = \theta$; and converse holds if Z_1, $Z_2 \in \mathfrak{L}^{so}(\Lambda(X),\mu(Y))$.

Proof:(i) From the definition of an o-matrix transformation, for some j in $\mathbb{N}$ and $\bar{x} = \delta^x_j$ in $\Lambda(X)$ with $x \in X$,

$$(Z(\delta^x_j))_i = (Z(\bar{x}))_i = \underset{n\to\infty}{o\text{-}\lim}\sum_{k=1}^{n} Z_{ik}(x_k) = Z_{ij}(x),\ \forall\ i\in \mathbb{N}.$$

Hence (i) follows.

(ii) If Z is a positive o-matrix transformation from $\Lambda(X)$ to $\mu(Y)$, then for each x in K, $\delta_j^x \geq \bar{\theta}$ in $\Lambda(X)$, for each $j\geq 1$ and so $Z_{ij}(x)=(Z(\delta_j^x))_i \geq \theta$, for each i,j in $\mathbb{N}$.

Conversely, if $\bar{x} \geq \bar{\theta}$ in $\Lambda(X)$, then $\sum_{j=1}^{n} Z_{ij}(x_j) \geq \theta$, for each n in $\mathbb{N}$ and all $i \in \mathbb{N}$, and thus $(Z(\bar{x}))_i = \underset{n\to\infty}{\text{o-lim}} \sum_{j=1}^{n} Z_{ij}(x_j) \geq \theta$ in Y, $\forall$ $i\geq 1$.

(iii) For proving the order boundedness of each Z_{ij}, consider an order bounded subset B of X. Then $B \subset [x,y]$, for some x,y in X. For given j in $\mathbb{N}$, write $\bar{B}=\{\delta_j^u : u \in B\}$. Then $\bar{B} \subset [\delta_j^x, \delta_j^y]$ in $\Lambda(X)$ and so there exist $\bar{a},\bar{b}$ in $\mu(Y)$ such that $Z(\bar{B}) \subset [\bar{a},\bar{b}]$, by our hypothesis. Now, applying (i), we get $Z_{ij}(B) \subset [a_i,b_i]$, for each i in $\mathbb{N}$. Thus Z_{ij}'s, $i,j \in \mathbb{N}$, are order bounded linear operators from X to Y.

(iv) In order to prove this statement, let us consider a sequence $\{x_n\}$ in X such that $x_n \xrightarrow{(o)} \theta$ in X. Then, by Proposition 3.2.7(iii) and our hypothesis of sequential order continuity of Z,

$$Z(\delta_j^x n) \xrightarrow{(o)} \bar{\theta} \text{ in } \mu(Y),\ \forall\ j \in \mathbb{N}.$$

$$\Rightarrow \quad (Z(\delta_j^x n))_i = Z_{ij}(x_n) \xrightarrow{(o)} \theta, \text{ as } n\to\infty, \text{ in } Y,\ \forall\ i,j \geq 1,$$

cf. Proposition 3.2.7(ii). Thus (iv) follows.

(v) It is an immediate consequence of (i).

(vi) From the definition of the infimum of two positive operators, cf. Theorem 1.2.20, for $x \in K$ and i,j in $\mathbb{N}$, we have

$$(+) \quad \begin{aligned}(Z_{ij}^1 \wedge Z_{ij}^2)(x) &= \inf\{ Z_{ij}^1(y)+Z_{ij}^2(z) : y,z \in K,\ y+z = x \} \\ &= (\inf\{ Z^1(\bar{y})+ Z^2(\bar{z}) : \bar{y},\bar{z} \geq \bar{\theta},\ \bar{y}+\bar{z} = \delta_j^x \})_i \\ &= ((Z^1 \wedge Z^2)(\delta_j^x))_i\end{aligned}$$

$$\Rightarrow \quad Z_{ij}^1 \wedge Z_{ij}^2 = \theta \text{ if } Z^1 \wedge Z^2 = \theta.$$

Conversely, let us assume that $Z_{ij}^1 \wedge Z_{ij}^2 = \theta$, for each

i,j in $\mathbb{N}$. Then for $\bar{x}$ in $\Lambda(X)$, with $\bar{x} = \{x_n\}$, $x_n \in K$, $\forall\ n \geq 1$, we have by (+)

$$
\begin{aligned}
& (Z^1_{ij} \wedge Z^2_{ij})(x_j) && = ((Z^1 \wedge Z^2)(\delta^x_j j))_i \ , \ \forall\ i,j \in \mathbb{N} \\
\Rightarrow\ & (Z_1 \wedge Z_2)(\delta_j{}^x j) && = \bar{\theta},\ \forall\ j \geq 1 \\
\Rightarrow\ & (Z_1 \wedge Z_2)(\bar{x}^{(n)}) && = (Z_1 \wedge Z_2)(\sum_{j=1}^{n} \delta^x_j j) = \bar{\theta},\ \forall\ n \geq 1 \\
\Rightarrow\ & (Z_1 \wedge Z_2)(\bar{x}) && = \underset{n \longrightarrow \infty}{\text{o-lim}}\ (Z_1 \wedge Z_2)(\bar{x}^{(n)}) \\
& && = \theta,
\end{aligned}
$$

by Proposition 3.2.7(i) and Theorem 1.2.20(ii). Hence $Z_1 \wedge Z_2 = \bar{\theta}$ on $\bar{K}$ and therefore on $\Lambda(X)$. Thus (vi) follows.■

A characterization of matrix representation of a linear map on OVVSS is its sequential order continuity, as proved in

Theorem 4.2.3: With X ,Y ,$\Lambda(X)$, $\mu(Y)$ as mentioned above, we have

(i) Every sequentially order continuous linear map from $\Lambda(X)$ to $\mu(Y)$ is an o-matrix transformation; and conversely

(ii) Z is sequentially order continuous, if $\Lambda(X)$ is monotone, $\mu(Y)$ a scalar valued sequence space, i.e., $Y = \mathbb{R}$, and $Z \equiv [Z_{ij}]$ an order bounded o-matrix transformation with Z_{ij}'s sequentially order continuous for each i,j in $\mathbb{N}$.

Proof:(i) Let Z be a sequentially order continuous linear map from $\Lambda(X)$ to $\mu(Y)$. For i,j in $\mathbb{N}$, define linear operators Z_{ij} from X to Y by the relations

$$Z_{ij}(x) = (Z(\delta^x_j))_i,\ x \in X.$$

We now show that $Z \equiv [Z_{ij}]$. Let us, therefore, consider $\bar{x} = \{x_j\}$ in $\Lambda(X)$. Then, by Proposition 3.2.7(i),(ii) and sequential order continuity of Z, we get

$$(Z(\bar{x}^{(n)}))_i \xrightarrow{(o)} (Z(\bar{x}))_i,\ \text{as } n \longrightarrow \infty,\ \forall i \geq 1.$$

But $(Z(\bar{x}^{(n)}))_i = (Z(\sum_{j=1}^{n} \delta^x_j j))_i = \sum_{j=1}^{n} Z_{ij}(x_j)$, $\forall\ n \geq 1$. Thus (i)

holds.

(ii) Let us first note that Z_{ij}'s are order bounded by Proposition 4.2.2(iii) and so $Z_{ij} \in X^{so}$, for each i,j in $\mathbb{N}$ by Proposition 1.2.23(i). Write

$$T^i = \{Z_{ij} : j \geq 1\}, \ i \geq 1.$$

Then $T^i \in \Lambda^{\beta}(X^{so})$, for each i in $\mathbb{N}$ as $\sum_{j=1}^{m} Z_{ij}(x_j) \xrightarrow{(o)} (Z(\bar{x}))_i$ in $\mathbb{R}$, $\forall$ $i \geq 1$ and $\bar{x}$ in $\Lambda(X)$. But $\Lambda^{\beta}(X^{so}) = \Lambda^{x}(X^{so})$ by Proposition 1.4.9; and hence applying Theorem 3.4.2 for $\{T^i\} \subset \Lambda^{x}(X^{so})$, we can find a sequence $\{\bar{f}_i\} \subset [\Lambda(X)]^{so}$ such that for $\bar{x} \in \Lambda(X)$,

$$(*) \qquad \bar{f}_i(\bar{x}) = \langle \bar{x}, T^i \rangle = \sum_{j=1}^{\infty} Z_{ij}(x_j), \ \forall \ i \geq 1 .$$

For proving the sequential order continuity of Z, let us now consider a sequence $\{\bar{y}^n\}$ and $\bar{y}$ in $\Lambda(X)$ such that $\bar{y}^n \xrightarrow{(o)} \bar{y}$, where $\bar{y}^n = \{y_j^n\}$, $n \geq 1$ and $\bar{y} = \{y_j\}$. Then, by (*)

$$\bar{f}_i(\bar{y}^n) = \sum_{j=1}^{\infty} Z_{ij}(y_j^n) \xrightarrow{(o)} \sum_{j=1}^{\infty} Z_{ij}(y_j) = \bar{f}_i(\bar{y}), \ \forall \ i \geq 1.$$

But $\sum_{j=1}^{\infty} Z_{ij}(y_j^n) = (Z(\bar{y}^n))_i$ and $\sum_{j=1}^{\infty} Z_{ij}(y_j) = (Z(\bar{y}))_i$, for $i \geq 1$, and so

$$(Z(\bar{y}^n))_i \xrightarrow{(o)} (Z(\bar{y}))_i, \ \forall \ i \geq 1.$$

As $\{ Z(\bar{y}^n) : n \geq 1 \}$ is an order bounded sequence in $\mu(Y)$, by Proposition 3.2.7(iii), it follows that $Z(\bar{y}^n) \xrightarrow{(o)} Z(\bar{y})$. This completes the proof.∎

Before we pass on to the next result concerning the matrix representation of the adjoint of a sequentially order continuous linear map, let us make some simple observations concerning adjoint maps in

Proposition 4.2.4: Let $\langle X, X^{so} \rangle$, $\langle Y, Y^{so} \rangle$ be dual pairs and $T: X \longrightarrow Y$ be a sequentially order continuous linear map. Then the adjoint map T^* of T (i) transforms Y^{so} into X^{so}; (ii) is positive if T is positive; and (iii) is an order bounded sequentially order

continuous linear map if T is order bounded.

Proof: (i) From the equality

(+) $\langle T(x),g\rangle = \langle x,T^*(g)\rangle$, $x \in X$, $g \in Y^{so}$,

it follows that $T^*(g) \in X^{so}$ whenever $g \in Y^{so}$, for if $\{x_n\} \subset X$, order converges to θ in X, then by (+), $(T^*(g))(x_n)=g(T(x_n)) \longrightarrow 0$ in $\mathbb{R}$, as $n\longrightarrow\infty$, by the sequential order continuity of T and $g \in Y^{so}$.

(ii) This part is immediate again by (+), for if $x \geq \theta$ in X and $g \geq \theta$ in Y^{so}, then $(T^*(g))(x) \geq \theta$, i.e., $T^*(g) \geq \theta$ in X^{so}.

(iii) If T is order bounded, then $T=T^+-T^-$, by Theorem 1.2.20(i) and so $T^*=(T^+)^*-(T^-)^*$, where $(T^+)^*$ and $(T^-)^*$ are positive linear mappings from Y^{so} into X^{so} by (ii). Consequently, $T^* \in \mathcal{L}^b(Y^{so},X^{so})$ whenever $T \in \mathcal{L}^b(X,Y)$; and hence it suffices to prove the sequential order continuity of T^* when T is positive. Let us therefore, consider a sequence $\{g_n\}$ in Y^{so} such that $g_n \downarrow \theta$ in Y^{so}. Then, for each $x \in X$, by (+)

$$(T^*(g_n))(x) \longrightarrow 0, \text{ as } n\longrightarrow\infty \text{ in } \mathbb{R}.$$

$$\Rightarrow \quad T^*(g_n) \downarrow \theta \text{ in } X^{so},$$

by Theorem 1.2.22(ii) as $T^*(g_n)\downarrow$ in X^{so}.■

Theorem 4.2.5: Let $\langle\Lambda(X),\Lambda^x(X^{so})\rangle$ and $\langle\mu(Y),\mu^x(Y^{so})\rangle$ be dual pairs defined corresponding to the dual pairs $\langle X,X^{so}\rangle$ and $\langle Y,Y^{so}\rangle$ respectively. If Z is an order bounded sequentially order continuous linear map from $\Lambda(X)$ to $\mu(Y)$, given by the matrix $[Z_{ij}]$ of linear maps $Z_{ij}:X\longrightarrow Y$, $i,j \in \mathbb{N}$, then the adjoint map Z^* of Z is an o-matrix transformation from $\mu^x(Y^{so})$ to $\Lambda^x(X^{so})$ such that $Z^* \equiv Z^{\perp} \equiv [Z^*_{ji}]$, where the adjoint maps Z^*_{ij} of Z_{ij} carry Y^{so} into X^{so}. Further, $Z^* \equiv Z^{\perp}$ is an order bounded sequentially order continuous linear map.

Proof: We first prove the result for positive sequentially order continuous linear map Z from $\Lambda(X)$ to $\mu(Y)$. By Proposition 4.2.2, Z_{ij}'s are positive sequentially order continuous linear maps, for each i,j in $\mathbb{N}$, and so Z^*_{ij}'s are positive sequentially order continuous linear maps from Y^{so} into X^{so} by Proposition 4.2.4. As Z is a sequentially order continuous linear map from $\Lambda(X)$ to $\mu(Y)$ and $[\Lambda(X)]^{so} \equiv \Lambda^x(X^{so})$, $[\mu(Y)]^{so} \equiv \mu^x(Y^{so})$, it follows by Proposition 4.2.4 that Z^* maps $\mu^x(Y^{so})$ into $\Lambda^x(X^{so})$ and is order bounded.

We now establish that $Z^* \equiv Z^{\perp}$. Let us, therefore, consider $\bar{g}$ in $\mu^x(Y^{so})$ such that $\bar{g} \geq \bar{\theta}$, where $\bar{g}=\{g_n\}$, $g_n \in Y^{so}$, for $n \geq 1$. As $Z^*_{ij} \geq \theta$, for each i,j in $\mathbb{N}$, $\{\sum_{j=1}^{k} Z^*_{ji}(g_j) : k \geq 1\}$ is an increasing sequence in X^{so}, $\forall\ i \geq 1$. Further, for $x \in X$ and $k \geq 1$

$$\begin{aligned}
\langle x, \sum_{j=1}^{k} Z^*_{ji}(g_j)\rangle &= \sum_{j=1}^{k} \langle Z_{ji}(x), g_j\rangle \ , \ \forall\ i \geq 1, \\
&= \sum_{j=1}^{k} \langle (Z(\delta_i^x))_j, g_j\rangle \ , \forall\ i \geq 1, \\
&\longrightarrow \langle Z(\delta_i^x), \bar{g}\rangle, \text{ as } k \longrightarrow \infty, \ \forall\ i \geq 1.
\end{aligned}$$

But

$$\langle Z(\delta_i^x), \bar{g}\rangle = \langle \delta_i^x, Z^*(\bar{g})\rangle = \langle x, (Z^*(\bar{g}))_i\rangle, \text{ for each } i \geq 1.$$

Hence

$$(+) \qquad \langle x, \sum_{j=1}^{k} Z^*_{ji}(g_j)\rangle \xrightarrow{(o)} \langle x, (Z^*(\bar{g}))_i\rangle, \text{ as } k \longrightarrow \infty, \ \forall i \geq 1$$

$$\Rightarrow \qquad \sum_{j=1}^{k} Z^*_{ji}(g_j) \xrightarrow{(o)} (Z^*(\bar{g}))_i, \ \forall\ i \geq 1,$$

cf. Theorem 1.2.22(ii). As any arbitrary $\bar{g}$ in $\mu^x(Y^{so})$ can be written as the difference of two positive linear functionals, we have $Z^* \equiv [Z^*_{ji}] \equiv Z^{\perp}$, for positive Z.

Let us now consider an arbitrary order bounded, sequentially order continuous linear map Z from $\Lambda(X)$ to $\mu(Y)$.

Then, from the proof of the Proposition 4.2.4(iii), $Z^*=(Z^+)^*-(Z^-)^*$. But Z^+ and Z^- being sequentially order continuous are o-matrix transformations and thus by the preceding paragraph and Proposition 4.2.2(v),(vi), $(Z^+)^*\equiv [(Z^+_{ji})^*]$, $(Z^-)^*\equiv [(Z^-_{ji})^*]$. Thus $Z^*\equiv [Z^*_{ji}]$. The last statement of this theorem is immediate from the above paragraphs.

Aliter: An alternative proof of the above theorem may be given by using Theorem 4.2.3. Indeed, as Z is an order bounded, sequentially order continuous linear map from $\Lambda(X)$ to $\mu(Y)$, by Proposition 4.2.4, Z^* is so, defined from $\mu^X(Y^{SO})$ to $\Lambda^X(X^{SO})$. Hence , by Theorem 4.2.3(i), Z^* is an o-matrix transformation from $\mu^X(Y^{SO})$ to $\Lambda^X(X^{SO})$. Let $Z^*\equiv [T_{ij}]$, $T_{ij}:Y^{SO}\longrightarrow X^{SO}$, $i,j\geq 1$. By Proposition 4.2.2(i), for $i,j \in \mathbb{N}$, $f \in Y^{SO}$ and $x \in X$,

$$\begin{aligned}\langle x,T_{ij}(f)\rangle = \langle x, (Z^*(\delta^f_j))_i\rangle &= \langle Z(\delta^x_i),\delta^f_j\rangle \\ &= \langle (Z(\delta^x_i))_j,f\rangle \\ &= \langle Z_{ji}(x),f\rangle \\ &= \langle x,Z^*_{ji}(f)\rangle\end{aligned}$$

$$\Rightarrow \qquad T_{ij} = Z^*_{ji},\ \forall i,j \geq 1. \text{ Thus } Z^*\equiv [Z^*_{ji}].\blacksquare$$

Converse of the above theorem is proved in

Theorem 4.2.6: Let X, X^{SO}, Y, Y^{SO}, $\Lambda(X)$, $\Lambda^X(X^{SO})$, $\mu(Y)$, $\mu^X(Y^{SO})$ be as in Theorem 4.2.5. If Z is a $\sigma(\Lambda(X),\Lambda^X(X^{SO}))$ – $\sigma(\mu(Y),\mu^X(Y^{SO}))$ continuous linear map from $\Lambda(X)$ to $\mu(Y)$ such that its adjoint map Z^* is an order bounded sequentially order continuous linear map from $\mu^X(Y^{SO})$ to $\Lambda^X(X^{SO})$, given by the matrix $[R_{ij}]$ of linear maps, R_{ij}: $Y^{SO}\longrightarrow X^{SO}$, then Z is order bounded, sequentially order continuous and is an o-matrix transformation from $\Lambda(X)$ to $\mu(Y)$ such that $Z \equiv [R^*_{ji}]$, where R^*_{ji}'s are the adjoint maps of R_{ji}'s from X to Y.

Proof: From the relation

(*) $\langle Z(\bar{x}),\bar{f}\rangle = \langle \bar{x},Z^*(\bar{f})\rangle,\ \forall\ \bar{x} \in \Lambda(X),\ \bar{f} \in \mu^x(Y^{so}),$

and the fact that Z^{**} takes $\Lambda(X)$ into $\mu(Y)$ by Proposition 1.3.4, it follows that $Z \equiv Z^{**}$. Now, invoking the proof of Proposition 4.2.4(iii), we get that Z is an order bounded linear map.

Further, for proving the sequential order continuity of Z, it suffices to consider the case when Z^* is positive. Let us, therefore consider a sequence $\{\bar{x}^n\} \subset \Lambda(X)$ with $\bar{x}^n \downarrow \bar{\theta}$ in $\Lambda(X)$. Then, for any $\bar{f} \in \mu^x(Y^{so})$, (*) yields

$$\langle Z(\bar{x}^n),|\bar{f}|\rangle \longrightarrow 0, \text{ as } n\to\infty.$$

i.e., $Z(\bar{x}^n) \longrightarrow \bar{\theta}$, in $|\sigma|(\Lambda(X),\Lambda^x(X^{so}))$ which is a Hausdorff l.s. topology. Hence by Proposition 1.3.9(iii) , $Z(\bar{x}^n) \downarrow \bar{\theta}$ in $\Lambda(X)$, i.e., Z is sequentially order continuous.

Now, applying Theorem 4.2.3(i), Z is an o-matrix transformation given by the matrix, say, $[A_{ij}]$. We claim that $A_{ij}=R^*_{ji}$, $\forall$ $i,j\in \mathbb{N}$. Indeed, for $x \in X$, $f \in Y^{so}$, by Proposition 4.2.2(i), we have

$$\begin{aligned} f(A_{ij}(x)) &= \langle (Z(\delta^x_j))_i,f\rangle = \langle Z(\delta^x_j),\delta^f_i\rangle \\ &= \langle \delta^x_j,Z^*(\delta^f_i)\rangle = \langle x,(Z^*(\delta^f_i))_j\rangle \\ &= \langle x,R_{ji}(f)\rangle = f(R^*_{ji}(x)) \end{aligned}$$

$$\Rightarrow \quad A_{ij} = R^*_{ji},\ \forall\ i,j \text{ in } \mathbb{N}.$$

Thus $Z \equiv [R_{ji}{}^*]$ and the result follows.■

Remark: An alternative proof for the sequential order continuity of positive Z in the above result runs as follows:

Let $\bar{x}^n \downarrow \bar{\theta}$ in $\Lambda(X)$, then $Z(\bar{x}^n) \downarrow$ in $\mu(Y)$ and if $\bar{y} \in \mu(Y)$ is such that $\bar{\theta} \le \bar{y} \le Z(\bar{x}^n)$, for each $n \ge 1$, then for any $\bar{g} \ge \bar{\theta}$ in $\mu^x(Y^{so})$, $\bar{g}(\bar{y}) \le \bar{g}(Z(\bar{x}^n)) = Z^*(\bar{g})(\bar{x}^n) \downarrow 0$ in $\mathbb{R}$ as $Z^*(\bar{g}) \in$

$\Lambda^x(X^{so})$. Hence $\bar{g}(\bar{y}) = 0$, for each $\bar{g}$ in $\mu^x(Y^{so})$. Consequently, $\bar{y} = \bar{\theta}$ in $\mu(Y)$ and so $Z(\bar{x}^n) \downarrow \bar{\theta}$ in $\mu(Y)$.

Lastly, in this section, we mention an application of the above results contained in

Theorem 4.2.7: Let X be an order complete Riesz space such that $\langle X,X^{so}\rangle$ forms a dual pair and $\sigma(X^{so},X)$-bounded sets are order bounded in X^{so}. Also, assume that $\Lambda(X)$ is an ideal of $\Omega(X)$ and is a simple VVSS for the topology $\sigma(\Lambda(X),\Lambda^x(X^{so}))$. For a matrix $Z \equiv [Z_{ij}]$ of positive linear maps Z_{ij}, $i,j \geq 1$, from X into itself, consider the following statements:

(i) Z is an order bounded, sequentially order continuous, o-matrix transformation from $\ell^1(X)$ to $\Lambda(X)$.

(ii) The adjoint Z^* of Z is an order bounded, sequentially order continuous o-matrix transformation from $\Lambda^x(X^{so})$ to $\ell^\infty(X^{so})$ such that $Z^* \equiv Z^\perp \equiv [Z^*_{ji}]$.

(iii) For $x \in X$, the sequence $\{y^j_x : j \geq 1\}$, where $y^j_x = \{Z_{ij}(x): i \geq 1\}$ is an order bounded set in $\Lambda(X)$.

Then, (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii). If Z is $\sigma(\ell^1(X),\ell^\infty(X^{so}))$-$\sigma(\Lambda(X),\Lambda^x(X^{so}))$ continuous, then (ii) $\Rightarrow$ (i). Further, if (iii) holds, then the linear transformation A: $\Lambda^x(X^{so}) \longrightarrow \Omega(X^{so})$, given by the matrix $[Z^*_{ji}]$, has its range contained in $\ell^\infty(X^{so})$.

Proof: (i) $\Rightarrow$ (ii). It is an immediate consequence of Theorem 4.2.5, since $(\ell^1(X))^x = \ell^\infty(X^{so})$ by Proposition 3.4.6(iv).

(i) $\Rightarrow$ (iii). For a given $j \in \mathbb{N}$ and $x \in K$, observe that $y^j_x = Z(\delta^x_j)$, by Proposition 4.2.2(i) and $\delta^x_j \in \ell^1(X)$. Thus $y^j_x \in \Lambda(X)$, $\forall j \geq 1$.

As $\Lambda(X)$ is simple, for proving the order boundedness of the set $\{y^j_x : j \geq 1\}$, for a given x in K, it suffices to show that

it is $\sigma(\Lambda(X),\Lambda^{x}(X^{so}))$-bounded. Therefore, consider a positive element $\bar{f}=\{f_j\}$ in $\Lambda^{x}(X^{so})$. As (i) $\Rightarrow$ (ii), $Z^{*}(\bar{f})\in \ell^{\infty}(X^{so})$ and so there exists g in X^{so} such that

$$\langle x,(Z^{*}(\bar{f}))_i\rangle \le g(x),\ \forall\ i\ge 1.$$

As $(Z^{*}(\bar{f}))_i = \text{o-}\lim_{n\to\infty} \sum_{j=1}^{n} Z_{ji}^{*}(f_j)$, $\forall\ i \ge 1$ and $\langle x, \sum_{j=1}^{n} Z_{ji}^{*}(f_j)\rangle = \sum_{j=1}^{n} f_j(Z_{ji}(x))$, $\forall\ n\ge 1$, we get by Theorem 1.2.22(i), that

$$\sum_{j=1}^{\infty} f_j(Z_{ji}(x)) \le g(x),\ \forall\ i\ge 1.$$

$$\Rightarrow \qquad \langle y_x^j,\bar{f}\rangle = \sum_{k=1}^{\infty} f_k(Z_{kj}(x)) \le g(x),\ \forall\ j \ge 1.$$

Thus $\{y_x^j\colon j\ge 1\}$ is $\sigma(\Lambda(X),\Lambda^{x}(X^{so}))$-bounded and so order bounded in $\Lambda(X)$. Hence, the results holds for any x in X.

The implication (ii) $\Rightarrow$ (i) when Z is restricted to be $\sigma(\ell^{1}(X),\ell^{\infty}(X^{so}))$- $\sigma(\Lambda(X),\Lambda^{x}(X^{so}))$ continuous, is immediate from Theorem 4.2.6.

For proving the last statement, we first show that for $\bar{f} \ge \bar{\theta}$ in $\Lambda^{x}(X^{so})$, $\text{o-}\lim_{n\to\infty} \{ \sum_{j=1}^{n} Z_{ji}^{*}(f_j)\}$ exists in X^{so} for each i in $\mathbb{N}$. Indeed, for $x \ge \theta$ in X, y_x^i is in $\Lambda(X)$ and therefore the inequality $\langle y_x^i,\bar{f}\rangle < \infty$ yields the pointwise convergence of the sequence $\{ \sum_{j=1}^{n} Z_{ji}^{*}(f_j)\}$ in X^{so}. We now apply Theorems 1.2.22(ii) and 1.2.20(ii) to get the required order limit.

Next, we prove that $A(\bar{f}) \in \ell^{\infty}(X^{so})$ for $\bar{f} \ge \bar{\theta}$ in $\Lambda^{x}(X^{so})$. For $x \ge \theta$ in X, choose $\bar{y} = \{y_i\}$ in $\Lambda(X)$ such that $y_x^j \le \bar{y}$, for every $j\ge 1$. Then

$$Z_{ij}(x) \le y_i,\ \forall i\ge 1,\ j\ge 1.$$

$$\Rightarrow \quad \langle x,A(\bar{f})_i\rangle = \sum_{j=1}^{\infty} (Z_{ji}^{*}(f_j))(x) = \sum_{j=1}^{\infty} f_j(Z_{ji}(x)) \le \sum_{j=1}^{\infty} f_j(y_j),\ \forall i\ge 1.$$

Hence, $\{(A(\bar{f}))_i : i\ge 1\}$ is $\sigma(X^{so},X)$-bounded subset of X^{so} and so it is order bounded by our hypothesis, i.e., $A(\bar{f}) \in \ell^{\infty}(X^{so})$. This

establishes the result completely. ∎

4.3 <u>Certain o-Matrix Transformations</u>: As mentioned in the introductory remarks of this chapter, this section is devoted to the characterizations of o-matrix transformations on certain OVVSS introduced in (3.4.4). For the derivation of some of these results, we make use of a general result mentioned in

Proposition 4.3.1: Let X be a Riesz space, Y an order complete Riesz space and $\{T_n\}$ a sequence of positive linear operators from X to Y. Then a necessary and sufficient condition for $\{\sum_{i=1}^{n} T_i(x_i)\}$ to order converge in Y whenever $\{x_n\}$ order converges in X, is that $\{\sum_{i=1}^{n} T_i\}$ order converges in $\mathcal{L}^b(X,Y)$.

Proof: To prove the necessary part, for the constant sequence $\{x\}$, $x \in X$, define a linear operator $T:X \longrightarrow Y$ as

$$T(x) = \underset{n\to\infty}{o\text{-}\lim}\ ((\sum_{i=1}^{n} T_i)(x))$$

Then, $\sum_{i=1}^{n} T_i \uparrow T$ in $\mathcal{L}^b(X,Y)$ by Theorem 1.2.22(ii).

In order to establish the sufficiency, consider an order converging sequence $\{x_n\}$ in X, with $|x_n| \le x$, $\forall\ n \ge 1$. Write

$$t_n = \sum_{i=1}^{n} T_i(x_i) \ ,\ n \in \mathbb{N}.$$

Then, for $n > m$,

$$|t_n - t_m| = |\sum_{i=m+1}^{n} T_i(x_i)| \le \sum_{i=m+1}^{n} T_i(x).$$

$\Rightarrow$ $\{t_n\}$ is an order Cauchy sequence in Y, as $\{\sum_{i=1}^{n} T_i(x)\}$, being order convergent is order Cauchy in Y. As Y is order complete, $\{t_n\}$ order converges in Y by Proposition 1.2.13(iii). ∎

Let us now begin with the characterization of an o-matrix transformation on the OVVSS c(X).

Proposition 4.3.2: Let X be an order complete Riesz space with

diagonal property and Z_{ij}'s, positive linear operators from X into itself. Then $Z \equiv [Z_{ij}]$ is an o-matrix transformation from c(X) to c(X) such that the transformed sequence order converges to the same limit if and only if

(i) $\sum_{j=1}^{n} Z_{ij} \xrightarrow{(o)} Z_i$, as $n \to \infty$, for some positive linear operator Z_i, $i \geq 1$;

(ii) $Z_i(x) \xrightarrow{(o)} x$, as $i \to \infty$, $\forall\, x \in X$; and

(iii) $Z_{ij}(x) \xrightarrow{(o)} \theta$, as $i \to \infty$, $\forall\, j \geq 1$ and $x \in X$.

Proof: We first derive the necessary conditions (i), (ii) and (iii).

(i) Since $\{\sum_{j=1}^{n} Z_{ij}(x_j)\}$, for each $\bar{x} = \{x_j\} \in c(X)$, order converges in X; by Proposition 4.3.1, the sequence $\{\sum_{j=1}^{n} Z_{ij} : j \geq 1\}$ of operators order converges to some positive linear operator Z_i in $\mathcal{L}^b(X)$ for each $i \in \mathbb{N}$.

(ii) For $x \in X$, consider the constant sequence $\{x_n\}$, $x_n = x$, $\forall\, n \geq 1$ in c(X). As

$$(Z(\bar{x}))_i = o\text{-}\lim_{n\to\infty} \sum_{j=1}^{n} Z_{ij}(x) = Z_i(x),\ \forall\, i \in \mathbb{N},$$

and $x_n \xrightarrow{(o)} x$ in X, it follows that $Z_i(x) \xrightarrow{(o)} x$, as $i \to \infty$, by our hypothesis.

(iii) For given $j_o \in \mathbb{N}$ and $x \in X$, we have

$$Z(\delta^{x}_{j_o}) = \{Z_{ij_o}(x),\ i \geq 1\}.$$

As $\delta^{x}_{j_o} \in c_o(X)$, $(Z(\delta^{x}_{j_o}))_i = Z_{ij_o}(x) \xrightarrow{(o)} \theta$, as $i \to \infty$, in X. As $j_o \in \mathbb{N}$ is arbitrary, we get (iii).

For proving the sufficiency, let us consider a sequence $\{x_j\}$ in c(X). Then by the preceding proposition and (i), $o\text{-}\lim_{n\to\infty} \sum_{j=1}^{n} Z_{ij}(x_j)$ exists in X, for each $i \geq 1$. Write

$$t_i = o\text{-}\lim_{n\to\infty} \sum_{j=1}^{n} Z_{ij}(x_j), \quad i \geq 1.$$

If $x_j \xrightarrow{(o)} x$ in X, then by Proposition 1.2.13(ii), for any $\varepsilon > 0$, we can find $j_o \in \mathbb{N}$ such that

$$|x_j - x| \leq \varepsilon u, \ \forall\ j \geq j_o,$$

where $u \in X$ is the regulator of convergence. By (ii), we can find $v \in X$ such that $\theta \leq Z_i(u) \leq v$, $\forall i\geq 1$. Then

$$\begin{aligned}
|t_i - x| = |(o\text{-}\lim_{n\to\infty} \sum_{j=1}^{n} Z_{ij}(x_j)) - x| &\leq |(o\text{-}\lim_{n\to\infty} \sum_{j=1}^{n} Z_{ij}(x_j)) - (o\text{-}\lim_{n\to\infty} \sum_{j=1}^{n} Z_{ij}(x))| \\
&\quad + |Z_i(x) - x| \\
&= |o\text{-}\lim_{n\to\infty} \sum_{j=1}^{n} Z_{ij}(x_j - x)| + |Z_i(x) - x| \\
&\leq |\sum_{j=1}^{j_o} Z_{ij}(x_j - x)| + |o\text{-}\lim_{n\to\infty} \sum_{j=j_o}^{n} Z_{ij}(x_j - x)| \\
&\quad + |Z_i(x) - x| \\
&\leq \sum_{j=1}^{j_o} Z_{ij}(|x_j - x|) + \sup_{j\geq j_o} Z_i(|x_j - x|) + |Z_i(x) - x| \\
&\leq \sum_{j=1}^{j_o} Z_{ij}(|x_j - x|) + \varepsilon v + |Z_i(x) - x|
\end{aligned}$$

$\Rightarrow$ $t_i \xrightarrow{(o)} x$, as $i \longrightarrow \infty$, since $\sum_{j=1}^{j_o} Z_{ij}(|x_j - x|) \xrightarrow{(o)} \theta$ by (iii), and $Z_i(x) \xrightarrow{(o)} x$ in X by (ii); and the order convergence coincides with the uniform convergence in X. The proof is now complete.∎

Remark: Let us note that a matrix $[Z_{ij}]$ of positive linear operators Z_{ij}'s from X into itself, satisfying the conditions (i), (ii) and (iii) of the above proposition, transforms $c_o(X)$ into $c_o(X)$.

For the space $\ell^\infty(X)$, we have the following characterization of the o-matrix transformation:

Theorem 4.3.3: Let X be an order complete Riesz space . A matrix $Z \equiv [Z_{ij}]$ of positive linear operators Z_{ij}'s, defined from X into

itself, is an o-matrix transformation from $\ell^{\infty}(X)$ to $\ell^{\infty}(X)$ if and only if

(i) $\{\sum_{j=1}^{n} Z_{ij} : n \geq 1\}$ order converges to some positive linear operator Z_i on X, for each $i \in \mathbb{N}$; and

(ii) the set $\{Z_i(x) : i \geq 1\}$ is an order bounded subset of X for each x in X.

Proof: Assume that Z is an o-matrix transformation from $\ell^{\infty}(X)$ into itself. Then, invoking the proof of the necessary part of Proposition 4.3.1, with T_j replaced by Z_{ij} and T by Z_i for given $i \in \mathbb{N}$, the condition (i) follows.

Further, the sequence $\{Z_i(x) : i \geq 1\}$ is the image of the constant sequence $\{x\}$ in $\ell^{\infty}(X)$, for each $x \in X$; and so (ii) follows.

For proving the sufficient part, consider an element $\bar{x} = \{x_j\}$ of $\ell^{\infty}(X)$ so that $|x_j| \leq x$, for each $j \in \mathbb{N}$ and for some $x \in X$. Then, for $n > m$ in $\mathbb{N}$

$$\left|\sum_{j=1}^{n} Z_{ij}(x_j) - \sum_{j=1}^{m} Z_{ij}(x_j)\right| = \left|\sum_{j=m+1}^{n} Z_{ij}(x_j)\right| \leq \sum_{j=m+1}^{n} Z_{ij}(x), \ \forall i \geq 1.$$

$\Rightarrow$ $\{\sum_{j=1}^{n} Z_{ij}(x_j)\}$ is order Cauchy as $\{\sum_{j=1}^{n} Z_{ij}(x)\}$ being order convergent, is order Cauchy in X, for each $i \in \mathbb{N}$. Hence, it is an order convergent sequence in X for each $i \in \mathbb{N}$ and $\bar{x} \in \ell^{\infty}(X)$, in view of Proposition 1.2.13(iii). Set

$$t_i = o\text{-}\lim_{n\to\infty} \sum_{j=1}^{n} Z_{ij}(x_j), \ i \geq 1.$$

Then, $\{t_i\} \in \ell^{\infty}(X)$ by (ii) and hence $Z \equiv [Z_{ij}]$ defines an o-matrix transformation from $\ell^{\infty}(X)$ into itself.■

In case of matrices of linear operators, transforming cs(X) to c(X), we have the following two results mentioning separately the sufficient and necessary conditions:

Theorem 4.3.4: Let X be an order complete Riesz space with diagonal property and $Z \equiv [Z_{ij}]$, a matrix of positive and sequentially order continuous linear operators satisfying the conditions

(i) $o\text{-}\lim\limits_{n\to\infty} \sum\limits_{j=1}^{n} |Z_{ij}-Z_{ij+1}| = Z_i$, for some positive and sequentially order continuous linear operators $Z_i \in \mathcal{L}^{so}(X), \forall\ i\geq 1$;

(ii) The sequence $\{Z_i : i\geq 1\}$ obtained in (i) is an order bounded sequence in $\mathcal{L}^{so}(X)$; and

(iii) $Z_{ij} \xrightarrow{(o)} I$, as $i\longrightarrow\infty$, in $\mathcal{L}^{so}(X)$, $\forall j\geq 1$, where I is the identity operator on X.

Then Z is an o-matrix transformation from $cs(X)$ to $c(X)$ such that the transformed sequence order converges to the sum of the original sequence.

Proof: We first prove the result for those elements $\bar{x}=\{x_j\}$ of $cs(x)$, which have the zero sum. Indeed, we show that if $o\text{-}\lim\limits_{n\to\infty} \sum\limits_{j=1}^{n} x_j = \theta$, for $\bar{x} \in cs(X)$, then $t_i = o\text{-}\lim\limits_{n\to\infty} \sum\limits_{j=1}^{n} Z_{ij}(x_j)$ exists, for each $i \in \mathbb{N}$, and $t_i \xrightarrow{(o)} \theta$ in X. Let us write

$$s_m = \sum_{j=1}^{m} x_j, \quad m \in \mathbb{N}.$$

Then $s_m \xrightarrow{(o)} \theta$ and so by Proposition 1.2.13(ii), for given $\varepsilon > 0$, we can find $j_o \in \mathbb{N}$ such that $|s_j| \leq \varepsilon u$, $\forall j\geq j_o$, where u is the regulator of convergence. Let $A, B \in \mathcal{L}^{so}(X)$ be such that $|Z_i| \leq A$, and $|Z_{i1}| \leq B$, $\forall i\geq 1$. Then, for i,j in $\mathbb{N}$, we have

$$\begin{aligned} |Z_{ij}| &\leq |Z_{ij}-Z_{ij-1}|+|Z_{ij-1}-Z_{ij-2}|+\ldots+|Z_{i2}-Z_{i1}|+|Z_{i1}| \\ &\leq Z_i+|Z_{i1}| \\ &\leq A+B. \end{aligned}$$

As

$$\sum_{j=m}^{m+p} Z_{ij}(x_j) = \sum_{j=m}^{m+p-1} (Z_{ij}-Z_{ij+1})(s_j) + Z_{im+p}(s_{m+p}) - Z_{im}(s_{m-1}), \quad i,m,p \in \mathbb{N},$$

we get for $i \in \mathbb{N}$ and $m,p \geq j_o$

$$\left|\sum_{j=m}^{m+p} Z_{ij}(x_j)\right| \leq \varepsilon\, A(u) + \varepsilon(A+B)(u) + \varepsilon(A+B)(u) = \varepsilon(3A+2B)(u),$$

$\Rightarrow$ $\{\sum_{j=1}^{m} Z_{ij}(x_j)\}$ is relatively uniformly Cauchy and hence it is an order convergent sequence in X, for each $i\geq 1$, cf. Proposition 1.2.13(i),(ii). Let us write

$$t_i = o\text{-}\lim_{n\to\infty} \sum_{j=1}^{n} Z_{ij}(x_j), \quad i\geq 1.$$

Since

$$\sum_{j=1}^{m} Z_{ij}(x_j) = \sum_{j=1}^{m-1} (Z_{ij}-Z_{ij+1})(s_j) + Z_{im}(s_m), \quad i,m \in \mathbb{N},$$

and

$$|Z_{im}(s_m)| \leq (A+B)(|s_m|), \text{ for each } i\geq 1;$$

where $s_m \xrightarrow{(o)} \theta$, we infer that

$$t_i = o\text{-}\lim_{n\to\infty} \sum_{j=1}^{n} (Z_{ij}-Z_{ij+1})(s_j), \quad i \geq 1.$$

Hence

$$|t_i| \leq \left|\sum_{j=1}^{j_o} (Z_{ij}-Z_{ij+1})(s_j)\right| + \left|o\text{-}\lim_{m\to\infty} \sum_{j=j_o}^{m} (Z_{ij}-Z_{ij+1})(s_j)\right|, \ \forall i\geq 1$$

$$\Rightarrow \quad |t_i| \leq \left|\sum_{j=1}^{j_o} (Z_{ij}-Z_{ij+1})(s_j)\right| + \sup_{j\geq j_o} Z_i(|s_j|), \ \forall\, i \geq 1$$

$$\Rightarrow \quad |t_i| \leq \left|\sum_{j=1}^{j_o} (Z_{ij}-Z_{ij+1})(s_j)\right| + \varepsilon\, A(u), \ \forall\, i \geq 1$$

$\Rightarrow$ $t_i \xrightarrow{(o)} \theta$ in X, by Proposition 1.2.13 and the fact that $Z_{ij}-Z_{ij+1} \xrightarrow{(o)} \theta$, for each $j \in \mathbb{N}$, which follows from (iii).

If $\bar{x} \in cs(X)$ is such that $\sum_{j=1}^{\infty} x_j = x$ where $x \neq \theta$, then we define a sequence $\{y_j\}$ in cs(X), with zero sum as follows:

$$y_1 = x_1 - x \quad ; \quad y_j = x_j, \ j \geq 2.$$

As

$$\sum_{j=1}^{m} Z_{ij}(x_j) = \sum_{j=1}^{m} Z_{ij}(y_j) + Z_{i1}(x), \ \forall\, m,i \in \mathbb{N},$$

it follows from the preceding paragraph that $o\text{-}\lim_{m\to\infty} \sum_{j=1}^{m} Z_{ij}(x_j)$

exists, say t_i, for each $i \in \mathbb{N}$ and $t_i \xrightarrow{(o)} x$ in X by (iii). Hence the result follows.■

The converse of the above theorem holds in the following form:

Theorem 4.3.5: Let X be an order complete Riesz space and $Z \equiv [Z_{ij}]$, a matrix of positive and sequentially order continuous linear operators from X into itself such that $Z_{ij} \geq Z_{ij+1}$, for every i,j in $\mathbb{N}$. Assume that Z transforms cs(X) to c(X) such that the transformed sequence order converges to the sum of the original sequence. Then

(i) for each $j \geq 1$ and $x \in X$, $Z_{ij}(x) \xrightarrow{(o)} x$, as $i \to \infty$;

(ii) for some $Z_i \in \mathcal{L}^{so}(X)$, $Z_{in} \xrightarrow{(o)} Z_i$, as $n \to \infty$, in $\mathcal{L}^{so}(X)$; and

(iii) for each $x \in X$, $\{Z_i(x)\}$ is an order bounded sequence in X.

Proof: (i) For $j \in \mathbb{N}$ and $x \in X$, define $\bar{x} = \delta_j^x$. Then $\bar{x} \in cs(X)$ with sum as x and so from the hypothesis,

$$Z_{ij}(x) = (Z(\delta_j^x))_i \xrightarrow{(o)} x, \text{ as } i \to \infty,$$

in X. Thus (i) follows.

(ii) For each i in $\mathbb{N}$ and x in K, we have $\theta \leq Z_{in}(x)\downarrow_n$ in X; and hence, from the order completeness of X, $Z_{in}(x) \downarrow_n v_i$, for some $v_i \in X$, for each $i \in \mathbb{N}$. Thus we can define linear operators Z_i's from X into itself by

$$Z_i(x) = \underset{n\to\infty}{\text{o-lim}}\; Z_{in}(x),\ x \in X,\ i \in \mathbb{N}.$$

Consequently, $Z_{in} \downarrow Z_i$, for each i in $\mathbb{N}$ by Theorem 1.2.22(ii); and so $\{Z_i : i \geq 1\} \subset \mathcal{L}^{so}(X)$, in view of Theorem 1.2.20(ii).

(iii) Since $\theta \leq Z_{in}(x) \leq Z_{i1}(x)$, for each $i,n \in \mathbb{N}$ and $Z_{in}(x) \xrightarrow{(o)} Z_i(x)$, as $n \to \infty$, $\forall i \geq 1$, in X by (ii), we have

$$\theta \le Z_i(x) \le Z_{i1}(x),\ \forall i \ge 1,\ x \in K.$$

Now we apply (i) to get (iii).■

Restricting further the linear operators Z_{ij} in the hypothesis of the above theorem, we have

Theorem 4.3.6: Let $Z \equiv [Z_{ij}]$ and X be as in Theorem 4.3.5. Assume further that $Z_{ij} \ge Z_{i+1j}$, $\forall i,j \in \mathbb{N}$. Then the matrix $A \equiv [A_{ij}]$ of operators A_{ij}'s, where $A_{ij} = Z_{ij} - Z_{ij+1}$, $i,j \in \mathbb{N}$, is an o-matrix transformation from c(X) to $c_o(X)$.

Proof: Observe that we have one-to-one and onto correspondence R between c(X) and cs(X), defined by $R(\bar{x}) = \bar{u}$, where $\bar{x} = \{x_j\} \in c(X)$ and $\bar{u} = \{u_j\}$ is given by

$$u_1 = x_1,\quad u_j = x_j - x_{j-1},\ j \ge 2.$$

Let us now consider a positive $\bar{x} = \{x_n\}$ in c(X) with $x_n \xrightarrow{(o)} x$ in X, for some $x \in X$. Then we can find $y_n \downarrow \theta$ in X with $|x_n - x| \le y_n$, for every n in $\mathbb{N}$, and so

$$|Z_{in+1}(x_n - x)| \le Z_{in+1}(|x_n - x|) \le Z_{in+1}(y_n) \le Z_{i1}(y_n),\ \forall i,n \in \mathbb{N}.$$

$$\Rightarrow \quad (+) \qquad Z_{in+1}(x_n - x) \xrightarrow{(o)} \theta,\ \forall\ i \in \mathbb{N}.$$

Observe that with $\bar{u}$ defined as above, $Z(\bar{u}) \in c(X)$ and $(Z(\bar{u}))_i \xrightarrow{(o)} x$ by the hypothesis. From the proof of part (ii) of the preceding theorem, $Z_{in}(x)\downarrow_n v_i$, for some $v_i \in X$ and for each $i \in \mathbb{N}$. Thus $Z_{in+1}(x_n) \xrightarrow{(o)} v_i$, for each $i \in \mathbb{N}$ by (+). Further, as Z transforms cs(X) to c(X), we have

$$\begin{aligned}
(Z(\bar{u}))_i &= \operatorname*{o-lim}_{n\to\infty} \sum_{j=1}^{n} Z_{ij}(u_j) \\
&= \operatorname*{o-lim}_{n\to\infty} \Big(\sum_{j=1}^{n} ((Z_{ij} - Z_{ij+1})(x_j)) + Z_{in+1}(x_n) \Big) \\
&= \operatorname*{o-lim}_{n\to\infty} \sum_{j=1}^{n} A_{ij}(x_j) + v_i .
\end{aligned}$$

Thus $s_i = \operatorname*{o-lim}_{n\to\infty} \sum_{j=1}^{n} A_{ij}(x_j)$ exists in X, for each $i \in \mathbb{N}$.

Next we show that $v_i \downarrow x$ in X. Clearly, $v_i \ge v_{i+1}$, $\forall i \ge 1$,

as $v_i = \inf_n Z_{in}(x)$, $\forall i \geq 1$ and $Z_{ij} \geq Z_{i+1j}$, $\forall i,j \in \mathbb{N}$. Also, from this restriction and the condition (i) of the preceding theorem, $x = \inf_i Z_{in}(x)$, $\forall n \geq 1$, i.e., $x \leq Z_{in}(x)$; and so $x \leq v_i$, $\forall\ i \geq 1$. Let $a \geq \theta$ in X be such that $a \leq v_i$, $\forall i \geq 1$. Then $a \leq Z_{in}(x)$, $\forall i,n \geq 1$ yield that $a \leq x$. Hence $v_i \downarrow x$ in X.

Since

$$s_i = o\text{-}\lim_{n\to\infty} \sum_{j=1}^{n} (Z_{ij} - Z_{ij+1})(x_j) = (Z(\bar{u}))_i - v_i,\ i \geq 1,$$

$s_i \xrightarrow{(o)} \theta$ in X from the above paragraphs. Thus $A(\bar{x}) = \{s_i\} \in c_o(X)$, for $\bar{x} \in c(X)$. This completes the proof.■

4.4 <u>o-Precompactness of o-matrix transformations</u>: Let X and Y be two Riesz spaces such that Y is an order complete l.c.s. Riesz space which is equipped with a Lebesgue topology S, generated by the family $\mathfrak{D}_Y$ of Riesz seminorms. In order to consider the o-precompactness of an o-matrix transformation in this section, we consider the domain space as a Riesz subspace $\Lambda(X)$ of $\Omega(X)$ and the range space as $\lambda(Y)$ as defined in (1.4.11), with the topology $T_{\lambda(Y)}$ generated by the family $\{Q_{\bar{\beta}}: \bar{\beta} \in \lambda^X,\ q \in \mathfrak{D}_Y\}$ of seminorms, where $Q_{\bar{\beta}}$ is defined as in Section 3.3. Thus $\lambda(Y)$, which is an ideal in $\Omega(Y)$, becomes an order complete l.c.s.Riesz space by Proposition 3.3.3. With these underlying assumptions, o-precompactness of an o-matrix transformation is characterized as follows:

Theorem 4.4.1: A positive o-matrix transformation $Z \equiv [Z_{ij}]$ from $\Lambda(X)$ to $\lambda(Y)$ is o-precompact if and only if each Z_{ij} is o-precompact.

Proof: For proving the necessity, consider Z_{ij} for arbitrarily fixed i,j in $\mathbb{N}$. Let us also fix $x \in K$ and $\varepsilon > 0$ arbitrarily. Write

U for the ε-neighbourhood of θ in Y given by the seminorm $q \in \mathfrak{D}_Y$, i.e., $U = \{y \in Y: q(y) < \varepsilon\}$. Choose $\bar{\beta} \in \lambda^X$ such that its i-th co-ordinate is non-zero. As $Z[-\delta_j^x, \delta_j^x]$ is precompact, for the neighbourhood $U_{\bar{\beta}} = \{\bar{y} \in \lambda(Y): Q_{\bar{\beta}}(\bar{y}) < \varepsilon|\beta_i|\}$ of zero, there exists a finite set $\bar{F}$ in $\lambda(Y)$ such that

$$Z[-\delta_j^x, \delta_j^x] \subset \bar{F} + U_{\bar{\beta}}$$

$$\Rightarrow \qquad Z_{ij}[-x, x] \subset F_i + U,$$

where $F_i = \{y_i : \bar{y} \in \bar{F}\}$ is a finite subset of Y. Thus Z_{ij} is o-precompact.

In order to establish the sufficiency, for $i, j \in \mathbb{N}$, define the linear operators $Z_j^i: \Lambda(X) \longrightarrow \lambda(Y)$ by

$$Z_j^i(\bar{x}) = \delta_i^{Z_{ij}(x_j)}, \quad \bar{x} \in \Lambda(X).$$

We now show that Z_j^i's are o-precompact. Indeed, for $\bar{x} \geq \bar{\theta}$ in $\Lambda(X)$ and any ε-neighbourhood $U_{\bar{\beta}} = \{\bar{y} \in \lambda(Y): Q_{\bar{\beta}}(\bar{y}) < \varepsilon\}$, $\bar{\beta} \in \lambda^X$ with $\beta_i \neq 0$ (the case when $\beta_i = 0$, trivially follows) of $\bar{\theta}$ in $\lambda(Y)$,

$$Z_j^i[-\bar{x}, \bar{x}] \subset \delta_i^F + U_{\bar{\beta}},$$

where $\delta_i^F = \{ \delta_i^f : f \in F\}$, F being the finite subset of Y obtained corresponding to the neighbourhood $U = \{ y \in Y : q(y) < \varepsilon/|\beta_i|\}$ of zero in Y for the precompact set $Z_{ij}[-x_j, x_j]$

Now, for $i, n \in \mathbb{N}$, define $Z_i^{(n)}: \Lambda(X) \longrightarrow \lambda(Y)$ by

$$(*) \qquad Z_i^{(n)}(\bar{x}) = \sum_{j=1}^{n} Z_j^i(\bar{x}) = \delta_i^{\sum_{j=1}^{n} Z_{ij}(x_j)}.$$

These operators, being finite sum of o-precompact operators, are clearly o-precompact. Moreover, by the definition of o-matrix transformation and Proposition 3.2.7(iii), it follows that $Z_i^{(n)} \xrightarrow{(o)} Z_i(\bar{x})$, as $n \longrightarrow \infty$, for each $i \geq 1$, where $Z_i: \Lambda(X) \longrightarrow \lambda(Y)$ is defined by

$$Z_i(\bar{x}) = \delta_i^{(Z(\bar{x}))_i}, \ \forall i \geq 1.$$

Since $(\lambda(Y),T_{\lambda(Y)})$ is a Lebesgue space, by Proposition 3.3.6, $Z_i^{(n)}(\bar{x}) \longrightarrow Z_i(\bar{x})$, $\forall$ $i\geq 1$, in $T_{\lambda(Y)}$.

In order to prove that each Z_i is an o-precompact operator, fix $i \in \mathbb{N}$, and consider an interval $[-\bar{x},\bar{x}]$, for $\bar{x} \geq \bar{\theta}$ in $\Lambda(X)$ and a neighbourhood $\bar{U}$ of $\bar{\theta}$ in $\lambda(Y)$. Then we can find another neighbourhood $\bar{V}$ of $\bar{\theta}$ in $\lambda(Y)$ with $\bar{V}+\bar{V} \subset \bar{U}$. Hence, there exists $n_o \in \mathbb{N}$ such that

$$Z_i^{(n)}(\bar{x})-Z_i(\bar{x}) \in \bar{V},\ \forall\ n\geq n_o.$$

Choose a finite set $\bar{A}$ in $\lambda(Y)$ such that

$$Z_i^{(n_o)}[-\bar{x},\bar{x}] \subset \bar{A}+\bar{V}.$$

Hence,

$$Z_i[-\bar{x},\bar{x}] \subset \bar{A} +\bar{U}.$$

$$\Rightarrow \qquad Z_i \text{ is o-precompact, } \forall\ i\geq 1.$$

As $i \in \mathbb{N}$ is arbitrary, each Z_i is o-precompact. Consequently, the linear operators $Z^{(n)}:\Lambda(X)\longrightarrow\lambda(Y)$, defined by

$$Z^{(n)}=\sum_{i=1}^{n} Z_i,\ n\geq 1,$$

are also o-precompact. As $Z^{(n)}(\bar{x}) \xrightarrow{(o)} Z(\bar{x})$ by Proposition 3.2.7(i) ,in $\lambda(Y)$ and the topology of $\lambda(Y)$ is Lebesgue, $Z^{(n)}(\bar{x}) \longrightarrow Z(\bar{x})$ in $T_{\lambda(Y)}$. Now, proceeding as above, there exist $n_1\in \mathbb{N}$ and a finite set $\bar{B}$ in $\lambda(Y)$ with

$$Z^{(n)}(\bar{x})-Z(\bar{x}) \in \bar{V},\ \forall\ n \geq n_1, \text{ and } Z^{(n_1)}[-\bar{x},\bar{x}] \subset \bar{B}+\bar{V}$$

$$\Rightarrow \qquad Z[-\bar{x},\bar{x}] \subset \bar{B} +\bar{U}.$$

Hence, Z is o-precompact. This completes the proof.∎

For the o-precompactness of order bounded o-matrix transformations, the above result leads to

Theorem 4.4.2: Let $Z\equiv[Z_{ij}]$ be an order bounded, sequentially order continuous o-matrix transformation from $\Lambda(X)$ to $\lambda(Y)$. Then Z is o-precompact if and only if each Z_{ij} is o-precompact.

Proof : By Proposition 4.2.2(v),(vi) and Theorem 4.2.3, we can write $Z = [Z_{ij}^{+}] - [Z_{ij}^{-}]$. As $\mathcal{L}_{op}(\Lambda(X),\lambda(Y))$ forms a band in $\mathcal{L}^{b}(\Lambda(X),\lambda(Y))$ in view of Proposition 1.3.18, the result follows by Theorem 4.4.1.∎

CHAPTER 5

ORDER POLAR TOPOLOGIES

5.1 An Overview: This chapter incorporates results on order polar topologies which are defined on ordered vector spaces, as an outcome of the duality relationship of two vector spaces as well as their ordering. Whereas, in Section 5.2, we study order polar topologies $\tau_c(X,Y)$ and $\tau_{so}(X,Y)$ which are the topologies of uniform convergence on solid, convex, $\sigma(Y,X)$ relatively-compact subsets of Y satisfying the conditions A_2 and A_1 respectively; we consider the topology of uniform convergence on quasi-order precompact sets in Section 5.3. Our study of these topologies in these sections have been supported by several examples and counter examples from the theory of sequence spaces.

5.2 Polar Topologies $\tau_c(X,Y)$ and $\tau_{so}(X,Y)$: Throughout this section, we assume that *the Riesz space* X *is such that* $\langle X,X^c\rangle$ *forms a dual pair*. As $X^c \subset X^{so} \subset X^b$, X is in duality with X^{so} and X^b as well. Recalling now the definitions of the conditions A_2 and A_1 for subsets of X^c and X^{so} from Definition 1.2.28, we introduce

Definition 5.2.1: Let $\langle X,Y\rangle$ be a dual pair of Riesz spaces. Then the topologies *$\tau_c(X,Y)$ and $\tau_{so}(X,Y)$* are defined respectively as the topologies of uniform convergence on members of ζ_c and ζ_{so}, where

ζ_c = collection of all solid, convex, $\sigma(Y,X)$-relatively compact subsets of Y, which satisfy the condition A_2; here Y is an ideal in X^c;

ζ_{so} = collection of all solid, convex,

$\sigma(Y,X)$-relatively compact subsets of Y, which satisfy the condition A_1; when Y is an ideal in X^{so}.

Concerning these topologies, we have

Theorem 5.2.2: $\tau_c(X,Y)$ (resp. $\tau_{so}(X,Y)$) on X is the finest locally convex solid topology compatible with the dual pair $\langle X,Y\rangle$, satisfying the Lebesgue (resp. the σ-Lebesgue) property.

Proof: We prove the result for $\tau_c(X,Y)$; as the result for $\tau_{so}(X,Y)$ would follow analogously on replacing X^c by X^{so}, nets by sequences and the condition A_2 by A_1.

Observe that the topology $\tau_c(X,Y)$ is generated by the family of Riesz seminorms $\{ p_A : A \in \zeta_c \}$, where

$$p_A(x) = \sup_{y \in A} |\langle x,y\rangle|, \quad x \in X,$$

and so it is a l.c.s. topology. Further, for each positive element y in Y, the interval $[-y,y]$, being $o(X,Y)$-equicontinuous subset of Y, is $\sigma(Y,X)$ - relatively compact by Alaoglu Bourbaki Theorem, cf. [86], p.201; and $[-y,y]$ is in ζ_c. Consequently, $o(X,Y) \subset \tau_c(X,Y)$. Also $\tau_c(X,Y) \subset \tau(X,Y)$ (Mackey topology). Hence $\tau_c(X,Y)$ is a Hausdorff l.c.s. topology compatible with the dual pair $\langle X,Y\rangle$; cf. Proposition 1.3.1.

In order to show that it is the finest locally convex solid topology with the Lebesgue property and compatible with $\langle X,Y\rangle$, consider another l.c.s. topology τ on X, having these properties. As solid hull of a convex set is convex; in view of Proposition 1.3.8, we may assume that the collection ζ_τ generating the topology τ is comprised of τ-equicontinuous, convex and solid sets. We now show that each A in ζ_τ satisfies A_2. Therefore, consider a set A in ζ_τ and a net $\{x_\alpha : \alpha \in \Lambda\}$ in X such that $x_\alpha \xrightarrow{(o)} \theta$ in X. Then, from the equicontinuity of A, we

can find a solid τ-neighbourhood U of θ such that $A \subset U^{\circ}$; and from the order convergence of $\{x_{\alpha}\}$ to θ, we get a net $\{y_{\alpha}\}$ with $y_{\alpha}\downarrow \theta$ in X, such that $|x_{\alpha}| \le y_{\alpha}$, for each α in Λ. As τ is Lebesgue, $y_{\alpha} \longrightarrow \theta$ in τ and so for any $\varepsilon > 0$, we can find $\alpha_{o} \in \Lambda$ such that $y_{\alpha} \in \varepsilon U$, $\forall\ \alpha \ge \alpha_{o}$. Hence

(*) $p_{A}(x_{\alpha}) = \sup\limits_{y\in A} |\langle x_{\alpha}, y\rangle| \le \sup\limits_{y\in A} \langle |x_{\alpha}|, |y|\rangle \le \sup\limits_{y\in A} \langle y_{\alpha}, |y|\rangle < \varepsilon, \forall \alpha \ge \alpha_{o}$.

Thus A satisfies the condition A_2. Further, A being equicontinuous, is $\sigma(Y,X)$ relatively compact. Consequently, $A \in \zeta_{c}$ and hence $\zeta_{\tau} \subset \zeta_{c}$. This completes the proof.∎

Restricting the spaces X^{c} and X^{so}, we have the following equality relations amongst various solid topologies.

Proposition 5.2.3: Let X be a Riesz space for which $\sigma(X^{c},X)$ (resp. $\sigma(X^{so},X)$)-bounded sets are order bounded in X^{c}(resp. X^{so}). Then $\tau_{c}(X,X^{c})$, $o(X,X^{c})$ and $\beta(X,X^{c})$ (resp. $\tau_{so}(X,X^{so})$, $o(X,X^{so})$, and $\beta(X,X^{so})$) coincide.

Proof: The proof is immediate by our hypothesis and the definitions of the topologies mentioned in the statement of the proposition.∎

In the following three examples, we respectively illustrate the space X where the condition," $\sigma(X^{c},X)$-bounded sets in X^{c} are order bounded" is satisfied; the indispensability of this condition for the equality $\beta(X,X^{c}) = \tau_{c}(X,X^{c})$ to hold good; and the condition is not a necessary one if the topologies coincide.

Example 5.2.4: Consider the space $X = \phi$. Then $X^{c}=X^{so}= w$; cf. Note 1.4.8. Now by Proposition 1.4.3, it follows that any $\sigma(w,\phi)$-bounded set is order bounded in w.

Example 5.2.5: Let $X = \ell^{\infty}$. Then, $(\ell^{\infty})^{c} = (\ell^{\infty})^{so} = (\ell^{\infty})^{x} = \ell^{1}$, by

Note 1.4.8. But we know that $\beta(\ell^\infty,\ell^1)$ is the norm topology of ℓ^∞, for which the dual is not ℓ^1, i.e., $(\ell^\infty,\beta(\ell^\infty,\ell^1))^* \neq \ell^1$, cf. [96] p.129, and so $\tau_c(\ell^\infty,\ell^1) \subsetneq \beta(\ell^\infty,\ell^1)$ by Proposition 5.2.2; indeed, $(\ell^\infty,\tau_c(\ell^\infty,\ell^1))^* = \ell^1$ and each member of ζ_c is $\sigma(\ell^1,\ell^\infty)$-bounded. Also, observe that the set $A = \{ (1/n)e^n : n \geq 1 \}$ is $\sigma(\ell^1,\ell^\infty)$ -bounded; but it is not order bounded in ℓ^1.

Example 5.2.6: Take the space X as ℓ^2, for which $(\ell^2)^c = (\ell^2)^{so} = (\ell^2)^x = \ell^2$, cf. Note 1.4.8. In the space ℓ^2, the set $A = \{(1/\sqrt{n})e^n : n \in \mathbb{N}\}$ is $\sigma(\ell^2,\ell^2)$-bounded, which is not order bounded. On the other hand, $\beta(\ell^2,\ell^2) \equiv \tau(\ell^2,\ell^2) \equiv \tau_o$ and so $(\ell^2)^b \equiv (\ell^2,\beta(\ell^2,\ell^2))^* = \ell^2$, cf. Proposition 1.3.14(iii). Hence $\tau(\ell^2,\ell^2)$ which is a locally solid topology satisfying A_2, coincides with $\tau_c(\ell^2,\ell^2)$, cf. Proposition 1.3.13 and Theorem 5.2.2. Thus $\tau_c(\ell^2,\ell^2) \equiv \beta(\ell^2,\ell^2)$.

For proving rest of the results of this section, we need restrict the Riesz space X as introduced in

Definition 5.2.7: A Riesz space X is said to be (i) an *o-space* if for a sequence $\{x_n\}$ in X, $x_n \xrightarrow{(o)} \theta$ in X whenever $f(x_n) \longrightarrow 0$, for each f in X^c; and (ii) a *σ-o space* if for any sequence $\{x_n\} \subset X$, $x_n \xrightarrow{(o)} \theta$ in X if $f(x_n) \longrightarrow 0$, for every f in X^{so}.

Clearly, every o-space is a σ-o space and these two concepts coincide when $X^{so} = X^c$. It would be *interesting to know examples of σ-o spaces which are not o-spaces.* However, illustrating o-spaces, we have

Example 5.2.8: Consider the Riesz space w for which $w^c = \phi = w^{so}$, by Note 1.4.8. Let $\{\bar{\alpha}^n : n \geq 1\}$ be a $\sigma(w,\phi)$-null sequence in w. Then $\alpha_i^n \longrightarrow 0$, as $n \longrightarrow \infty$, for each $i \geq 1$, and also we can find an element $\{\beta_i\}$ in w such that

$$|\alpha_i^n| \le \beta_i, \ \forall n\ge 1,$$

cf. Proposition 1.4.3. Now, applying Proposition 3.2.7(iii), we get $\bar{\alpha}_n \xrightarrow{(o)} \bar{\theta}$. Thus w is a σ-o as well as an o-space.

Example 5.2.9: The space ℓ^∞ which is an ideal in w, is an o - as well as a σ-o-space, for $(\ell^\infty)^c = (\ell^\infty)^{so} = \ell^1$ by Note 1.4.8; and any $\sigma(\ell^\infty,\ell^1)$-null sequence $\{\bar{\alpha}^n\}$ in ℓ^∞, satisfies the relations

$$\alpha_i^n \longrightarrow 0, \ \forall\ i \in \mathbb{N}; \text{ and } |\alpha_i^n| \le M, \ \forall\ i,n \text{ in } \mathbb{N};$$

cf. [96], p.107, for the latter inequalities. Now applying Proposition 3.2.7(iii), we get the desired conclusion.

The spaces considered in the following three examples are not σ-o spaces and so they are not o-spaces.

Example 5.2.10: Consider the Riesz space X = C[0,1], with the usual pointwise ordering. It is known that $X^{so} = X^c = \{\theta\}$, for this space, cf.[246],p.153. Thus, in this space, every non-zero constant sequence $\sigma(X,X^c)$-converges to zero, but it does not order converge to zero.

Example 5.2.11: Let us take the Riesz space $X = \ell^1$, for which $(\ell^1)^c = (\ell^1)^{so} = (\ell^1)^x = \ell^\infty$ by Note 1.4.8. The sequence $\{(1/n)e^n : n\ge 1\}$ in ℓ^1 is not order convergent to zero as it is not order bounded; but for any $\bar{\alpha} \in \ell^\infty$, $\bar{\alpha}((1/n)e^n) = \alpha_n/n \longrightarrow 0$, as $n\longrightarrow\infty$, in $\mathbb{R}$. Thus $\{(1/n)e^n\}$ is a $\sigma(\ell^1,\ell^\infty)$-null sequence. Hence ℓ^1 is not an o-space.

Example 5.2.12: Here we consider the space $X = L^1[0,1]$ formed by the equivalence classes of Lebesgue integrable almost everywhere equal functions, defined over [0,1]. It is a well known fact that $L^1[0,1]$ is an order complete Riesz space for the ordering defined as , $[f] \le [g]$ in $L^1[0,1]$ if and only if $f(x) \le g(x)$ almost everywhere; cf.[6],p.71. Since the space $L^1[0,1]$, equipped with

the norm topology is a Banach lattice which is an abstract L-space (cf. Definition 1.3.19(ii)), it is a Lebesgue and hence a σ-Lebesgue space by Proposition 1.3.20(i). Consequently,

(*) $L^\infty=(L^1[0,1])^*\subset(L^1[0,1])^c\subset(L^1[0,1])^{so}\subset(L^1[0,1])^b=(L^1[0,1])^*=L^\infty$,

where L^∞ is the space of equivalence classes of essentially bounded functions on $[0,1]$, cf. Proposition 1.3.20(ii).

For constructing the required non-o-convergent null-sequence $\{[f_n]\}$ in $L^1[0,1]$, let us recall the sets E_n's used in the construction of the Cantor set, namely

$E_1= [0,1/3] \cup [2/3,1]$;

$E_2= [0,1/9] \cup [2/9,3/9] \cup [6/9,7/9] \cup [8/9,1]$;

. . . .

. . . .

$E_n= [0,1/(3^n)] \cup [2/(3^n),3/(3^n)] \cup \ldots \cup [(3^n-1)/(3^n),1]$;

. . . .

. . . .

If P is the Cantor set, write

$A_o=P$; $A_1=E_1^c$ and $A_n=(A_{n-1}\cup E_n)^c$, $n \geq 2$.

Then we have

(i) $[0,1]= \bigcup_{i\geq 0} A_i$;

(ii) $\mu(A_o)=0$ and $\mu(A_n)=(2^{n-1})/(3^n)$, $n \geq 1$; and

(iii) $A_i\cap A_j= \phi$, $i\neq j$.

Now define the sequence $\{f_n:n\geq 0\}$ as follows:

$f_o= \chi_{A_o}$; and $f_n=((3^n)/(n\ 2^n))\chi_{A_n}$, $n\geq 1$.

Then from (*), we have for $[\phi] \in L^\infty$,

$$|\langle[f_n],[\phi]\rangle| = |\int_0^1 f_n(x)\phi(x)dx|$$

$$= |\int_{\cup A_i} f_n(x)\phi(x)dx|$$

$$= |\int_{A_n}((3^n)/(n\ 2^n))\phi(x)dx|$$

$$\leq \left|\int_{A_n\cap B}((3^n)/(n\ 2^n))\phi(x)dx\right|+\left|\int_{A_n\cap B^c}((3^n)/(n\ 2^n))\phi(x)dx\right|,$$

where B is the set on which ϕ is bounded and $\mu(B^c)=0$. If M is the essential bound of ϕ, then

(+) $\qquad |\langle [f_n],[\phi]\rangle| \leq M/n.$

Thus $\{[f_n]\}$ is a $\sigma(L^1,L^\infty)$-null sequence. On the other hand, it is not order bounded in $L^1[0,1]$, as for any $[g] \in L^1[0,1]$ with $\theta \leq [f_n] \leq [g]$, $\forall\ n \geq 1$, we have $[f] \leq [g]$, where f on $[0,1]$ is defined as

$$f(x) = f_n(x),\ x \in A_n,\ n \geq 0.$$

But for $[f]$ to be an element of $L^1[0,1]$, $\int_0^1 f(x)dx < \infty$; however,

$$\begin{aligned}\int_0^1 f(x)dx = \int_{\cup A_i} f(x)dx &= \sum_{i\geq 0}\int_{A_i} f_i(x)dx \\ &= \sum_{i\geq 0}\int_{A_i}((3^i)/(i\ 2^i))dx \\ &= \sum_{i\geq 0}((3^i)/(i\ 2^i))((2^{i-1})/(3^i)) \\ &= (1/2)\sum_{i\geq 0} 1/i \\ &= \infty.\end{aligned}$$

This contradicts that $[f] \leq [g]$, i.e., the sequence $\{[f_n]:n \geq 0\}$ is not order bounded and so can not order converge to θ in $L^1[0,1]$. Hence $L^1[0,1]$ is not an o-space.

It is obvious that every $\tau_c(X,X^c)$ (resp. $\tau_{so}(X,X^{so})$)-closed set in X is σ-order closed since $\tau_c(X,X^c)$ (resp. $\tau_{so}(X,X^{so})$) is a Lebesgue (resp. σ-Lebesgue) topology. However, for o-spaces, we have

Proposition 5.2.13: Let X be an o-space. Then a σ-order closed set in X is $\tau_c(X,X^c)$-sequentially closed.

Proof: Let B be a σ-order closed set and $\{x_n\} \subset B$ is such that $x_n \longrightarrow x$ in $\tau_c(X,X^c)$. Then by Proposition 5.2.2, $f(x_n-x) \longrightarrow 0$, as

$n \to \infty$, for each f in X^c. Hence $x_n \xrightarrow{(o)} x$ in X by our hypothesis. Consequently, $x \in B$ as B is σ-order closed set. Thus the result follows.■

In case of σ-o spaces, we have

Proposition 5.2.14: For a σ-o space, the $\tau_{so}(X,X^{so})$-sequentially closed sets and σ-order closed sets are the same.

Proof: Proceeding on lines similar to that of the preceding proof, we can establish that every σ-order closed is $\tau_{so}(X,X^{so})$-sequentially closed; however, the converse is given in the paragraph preceding the above proposition.■

A characterization of a sequentially continuous linear operator on $(X,\tau_c(X,X^c))$ in terms of its order continuity is contained in

Proposition 5.2.15: Let X be an o-space. A linear operator T from $(X,\tau_c(X,X^c))$ into itself is sequentially continuous if and only if it is sequentially order continuous.

Proof: Let $T:X \to X$ be $\tau_c(X,X^c)$ - $\tau_c(X,X^c)$ sequentially continuous linear operator and $\{x_n\} \subset X$ be such that $x_n \xrightarrow{(o)} \theta$ in X. As $\tau_c(X,X^c)$ is a Lebesgue topology, $x_n \to \theta$ in $\tau_c(X,X^c)$. Hence $T(x_n) \to \theta$ in $\tau_c(X,X^c)$ and so $f(T(x_n)) \to 0$, as $n \to \infty$, for each f in X^c by Proposition 5.2.2. Since X is an o-space, $T(x_n) \xrightarrow{(o)} \theta$ in X. Thus T is sequentially order continuous. Conversely, if T is sequentially order continuous, then starting with a sequence $\{x_n\}$ converging to θ in $\tau_c(X,X^c)$, we have $x_n \xrightarrow{(o)} \theta$ and so $T(x_n) \xrightarrow{(o)} \theta$ in X. Now using the Lebesgue property of $\tau_c(X,X^c)$, we have $T(x_n) \to \theta$ in $\tau_c(X,X^c)$ and thus T is $\tau_c(X,X^c)$- $\tau_c(X,X^c)$ sequentially continuous.■

Replacing o-spaces by σ-o spaces in the above

proposition, we have the analogous result for the topology $\tau_{so}(X,X^{so})$ contained in

Proposition 5.2.16: If X is a σ-o-space, then a linear operator T from $(X,\tau_{so}(X,X^{so}))$ into itself is sequentially continuous if and only if it is sequentially order continuous.

Proof: The proof is omitted as it is analogous to the proof of the above result.■

Remark: We illustrate in Example 5.2.18, that the restriction of o-spaces in the above proposition is indispensable. However, for this example we need a general result on matrix transformations defined on real sequence spaces, proved in

Proposition 5.2.17: Let an ideal λ in w, equipped with a locally convex solid topology τ, be a Fréchét AK-space. Then every matrix transformation on λ is $\tau_c(\lambda,\lambda^c)$- $\tau_c(\lambda,\lambda^c)$ continuous.

Proof: In view of Propositions 1.3.14(iii), 1.4.4, 1.4.9 and the Note 1.4.8, we have

$$\lambda^x \equiv \lambda^{so} \equiv \lambda^c \equiv \lambda^b.$$

Hence, $\tau(\lambda,\lambda^b) \equiv \tau(\lambda,\lambda^c)$ is a locally solid topology satisfying the condition A_2, cf. Proposition 1.3.13. Consequently, by Proposition 5.2.2, $\tau_c(\lambda,\lambda^c) \equiv \tau(\lambda,\lambda^c)$. Thus applying Propositions 1.3.4, 1.4.6, we infer that any matrix transformation on λ is $\tau_c(\lambda,\lambda^c)$- $\tau_c(\lambda,\lambda^c)$ continuous.■

Let us quote from [175], the following

Example 5.2.18: Consider the space ℓ^2 and the operator $T:\ell^2 \longrightarrow \ell^2$, given by the matrix $[a_{pq}]$, where

$$a_{pq} = \begin{cases} 1/pq, & p \neq q \\ 0 & , \ p = q. \end{cases}$$

As ℓ^2, equipped with its norm topology, is a Fréchét AK-space, T is $\tau_c(\ell^2,\ell^2)$- $\tau_c(\ell^2,\ell^2)$ continuous by Proposition 5.2.17; but T

is not order continuous, cf. [175],p.170. Note that ℓ^2 is not an o-space; for the sequence $\{(1/\sqrt{n})e^n : n \geq 1\}$ is a $\sigma(\ell^2,\ell^2)$-null sequence, which is not order bounded in ℓ^2.

5.3 Topology of Uniform Convergence on Quasi-Order Precompact Sets

In the final section of this chapter, we consider the order polar topology $T_{op}(X^*,X)$ defined on the dual X^* of a locally convex solid Riesz space X as

Definition 5.3.1: For a l.c.s.Riesz space (X,τ), let ζ_p denote the collection of all convex, solid and quasi-order precompact subsets of (X,τ). Then the topology $T_{op}(X^*,X)$ *is the topology of uniform convergence on members of* ζ_p ; and hence it is generated by the family $\{q_A : A \in \zeta_p\}$ of seminorms, where

$$q_A(f) = \sup_{x \in A} |\langle x,f \rangle| \ .$$

Since the convex, solid hull of a quasi-order precompact set is quasi-order precompact [48], we may replace the collection ζ_p by ζ_h consisting of convex solid hulls of quasi-order precompact subsets of (X,τ). We now prove

Theorem 5.3.2: On τ-equicontinuous subsets of X^*, $T_{op}(X^*,X)$ coincides with the absolute weak topology $|\sigma|(X^*,X)$.

Proof: As the intervals $[-x,x]$, $x \in K$, are convex, solid and quasi-order precompact sets, it follows that $|\sigma|(X^*,X) = o(X^*,X) \subset T_{op}(X^*,X)$. Therefore, for establishing the result, it suffices to prove that $T_{op}(X^*,X)|_M \subset |\sigma|(X^*,X)|_M$, for any equicontinuous subset M of X^*, which we may assume to be a solid set in view of Proposition 1.3.8. Thus $\theta \in M$. Now, consider a net $\{f_\alpha : \alpha \in \Lambda\}$ in M such that $f_\alpha \longrightarrow \theta$ in $|\sigma|(X^*,X)$ and choose a τ-neighbourhood U of θ in X such that $M \subset U^\circ$. Then for any given $\varepsilon > 0$ and for a member A of ζ_p , there exists $x \geq \theta$ in X such that

$$A \subseteq [-x,x] + \varepsilon U.$$

Let $y \in A$, then $y = y_1 + y_2$, where $y_1 \in [-x,x]$ and $y_2 \in \varepsilon U$. Hence

$$|\langle y,f_\alpha\rangle| \le \langle |y_1|,|f_\alpha|\rangle + \langle |y_2|,|f_\alpha|\rangle$$
$$\le \langle x,|f_\alpha|\rangle + \varepsilon$$
$$\Rightarrow \quad q_A(f_\alpha) \le \langle x,|f_\alpha|\rangle + \varepsilon.$$

But $\langle x,|f_\alpha|\rangle \longrightarrow 0$ yields that $q_A(f_\alpha) \longrightarrow 0$. Consequently, $f_\alpha \longrightarrow 0$ in $T_{op}(X^*,X)|_M$;and this completes the proof.■

It is known that [48] a precompact subset of a locally solid Riesz space is always quasi-order precompact, but the converse is not necessarily true; for instance, consider

Example 5.3.3: Let $X = \ell^\infty$ be equipped with its norm topology. Then the order interval $[-e,e]$ in ℓ^∞ is clearly a quasi-order precompact set which is not a precompact subset of ℓ^∞; (indeed, $e^n \in [-e,e], \forall\ n \ge 1$).

However, the converse holds in the following form

Proposition 5.3.4: Let (X,τ) be a l.c.s. Riesz space such that $\sigma(X^*,X)=|\sigma|(X^*,X)$. Then any quasi-order precompact subset of X is precompact.

Proof: Let us consider a quasi-order precompact subset A of (X,τ) and a solid τ-neighbourhood U of θ. Then we can find a solid neighbourhood V of θ such that $V+V \subseteq U$; and a positive element x in X such that $A \subseteq [-x,x]+V$. Now, $[-x,x]$, being order bounded is topologically bounded subset of X, and so it is precompact by Proposition 1.3.14(ii) and our hypothesis $\sigma(X^*,X)=|\sigma|(X^*,X)$; cf. [196], p.38. Hence, there exists a finite set F in X such that $[-x,x] \subset F + V$. Consequently, $A \subset F + U$, i.e.,A is precompact.■

Note: As an example of a Riesz space for which $\sigma(X^*,X)=|\sigma|(X^*,X)$, we have the space w equipped with the absolute weak topology

$|\sigma|(w,\phi)$; indeed, $(w,|\sigma|(w,\phi))^{*}= \phi$ and $\eta(w,\phi)=|\sigma|(w,\phi)$ yields that $\sigma(w,\phi)= |\sigma|(w,\phi)$, cf. [96], p.106.

Finally, making use of the Diéudonné Theorem, we prove the last result of this chapter, contained in

Theorem 5.3.5: Let (X,τ) be a metrizable l.c.s.Riesz space satisfying the condition $\sigma(X^{*},X)=|\sigma|(X^{*},X)$. Then the topology $T_{op}(X^{*},X)$ is the finest locally convex topology on X^{*}, which induces on every equicontinuous subset of X^{*}, the same topology as $|\sigma|(X^{*},X)$.

Proof: As each quasi-order precompact set is precompact in X by Proposition 5.3.4, $T_{op}(X^{*},X)= \lambda(X^{*},X)$. Since $\sigma(X^{*},X)= |\sigma|(X^{*},X)$, applying Theorem 1.3.3, we get the desired conclusion.∎

CHAPTER 6

ORDER DECOMPOSITIONS, ASSOCIATED o-SEQUENCE SPACES AND o-SIMILARITY

6.1 An Overview: Our aim in this chapter is to carry out investigations on countable order decompositions defined in order-complete Riesz spaces, a notion introduced by Pinsker [184] in 1945, for arbitrary indexing set. Indeed, our study has been motivated by the theory of Schauder decomposition in locally convex spaces; cf. [190] and several references given therein. In Section 6.2 of this chapter, after having proved the existence of a countable order decomposition in the sequential order dual of an order complete Riesz space with a countable order decomposition, we derive sufficient conditions for the order perfectness of a Riesz space. Also, we justify our study of countable order decompositions with the help of an example of ordered vector valued sequence space containing a countable order decomposition. In the next section, the presence of countable order decompositions in a Riesz space as well as in its sequential order dual, helps us to introduce vector valued sequence spaces which form a dual pair. These results are used to characterize the o-similarity between two countable order decompositions in the final section.

6.2 Order Decompositions: Throughout this section, *we assume that* X *denotes an order complete Riesz space*. Recalling the definition of the projection operator $P \equiv P_B$ from X onto a band B, from Chapter 1, cf.Definition 1.2.25, consider the maps P^* and P^{**} defined by the relations

(6.2.1) $\qquad \langle x, P^*(f)\rangle = \langle P(x), f\rangle,\ x \in X,\ f \in B^{so};$

(6.2.2) $\langle f, P^{**}(\phi)\rangle = \langle P^*(f),\phi\rangle$, $f \in B^{so}$, $\phi \in (X^{so})^{so}$.

Concerning these maps, we have

Proposition 6.2.3: The maps P^* and P^{**} take B^{so} and $(X^{so})^{so}$ to X^{so} and $(B^{so})^{so}$ respectively and they are positive sequentially order continuous linear operators.

Proof: Firstly, we prove the result for P^*. Let $f \in B^{so}$ be a positive element. Then by Proposition 1.2.26 and (6.2.1), $P^*(f)$ is a positive map on X. Thus, in view of Proposition 1.2.20(iii), for proving the sequential order continuity of $P^*(f)$, it suffices to establish that $P^*(f)(x_n) \downarrow \theta$ whenever $x_n \downarrow \theta$ in X. Now, if $x_n \downarrow \theta$ in X, then $P(x_n) \downarrow \theta$ in B by Proposition 1.2.26. Hence by our choice of f and the relation (6.2.1), $P^*(f) \in X^{so}$. Thus P^* is a positive map from B^{so} to X^{so}.

To prove that P^* is sequentially order continuous, let us consider a sequence $\{f_k\} \subset B^{so}$ such that $f_k \downarrow \theta$ in B^{so}. Then $P^*(f_k) \downarrow$ in X^{so}, and from (6.2.1), $\langle x,P^*(f_k)\rangle \longrightarrow 0$, for any x in X. Consequently, by Theorem 1.2.22(ii), $P^*(f_k) \downarrow \theta$ in X^{so}.

Now using the above fact and the relation (6.2.2), we note that $P^{**}(\phi) \in (B^{so})^{so}$ for $\phi \in (X^{so})^{so}$ and P^{**} is a positive map on $(X^{so})^{so}$. Further, proceeding on lines similar to the above proof, we get the sequential order continuity of P^{**}.■

The rest of this section incorporates results *for an order complete Riesz space with a countable order decomposition* $\{M_n,P_n\}$ *formed by the family* $\{M_n : n \geq 1\}$ *of bands in* X *and the projection maps* $P_n \equiv P_{M_n}$ *from* X *onto* M_n, $n \geq 1$; cf. Definition 1.2.17. As P_n's are the projection maps from X into itself, by the above proposition, $P_n^*(X^{so})$ are linear subspaces of X^{so}, for $n \geq 1$ and *we write* $M_n^* = P_n^*(X^{so})$ for n in $\mathbb{N}$. Then, these subspaces

of X^{so} satisfy the conditions stated in

Proposition 6.2.4: (i) M_n^*'s are bands in X^{so};

(ii) M_n^*'s are pairwise disjoint;

(iii) the collection $\{ M_n^* : n \geq 1\}$ is complete in X^{so}; and

(iv) each M_n^* is a projection band.

Proof: (i) We first prove that each M_n^* is an ideal in X^{so}. Therefore, fix n in $\mathbb{N}$ arbitrarily and consider $f, g \in X^{so}$ such that $|f| \leq |P_n^*(g)|$. Then $|f| \leq P_n^*(|g|)$ by Theorem 1.2.20(i) and the positive character of P_n^*. Hence, for $m \neq n$ and $x \in K$, we have by (6.2.1),

(+) $\langle x, P_m^*(|f|)\rangle = \langle P_m(x), |f|\rangle \leq \langle P_m(x), P_n^*(|g|)\rangle = \langle P_n P_m(x), |g|\rangle$.

But $P_n(P_m(x)) \in M_n \cap M_m$, as $P_n(X) = M_n$ and $|P_n(P_m(x))| \leq |P_m(x)|$, by Proposition 1.2.26 and the fact that M_m is an ideal of X. Since $M_n \cap M_m = \{\theta\}$ by our hypothesis, $P_n(P_m(x)) = \theta$. Consequently, from (+), $P_m^*(|f|) = \theta$, i.e., $P_m^*(f) = \theta$ in X^{so} for any $m \neq n$. In order to show that $f = P_n^*(f)$, observe that $\sum_{i=1}^{k} P_i(x) \xrightarrow{(o)} x$, for any x in X by Theorem 1.2.18. Hence

$$f(x) = o\text{-}\lim_{k\to\infty} f\Big(\sum_{i=1}^{k} P_i(x)\Big) = o\text{-}\lim_{k\to\infty} \sum_{i=1}^{k} (P_i^*(f))(x) = (P_n^*(f))(x),\ \forall x \in X.$$

$$\Rightarrow \quad P_n^*(f) = f,\ \text{i.e., } f \in M_n^*.$$

For proving the order-closedness of M_n^*, in view of Proposition 1.2.10(ii), it suffices to consider a net $\{ f_\alpha : \alpha \in \Lambda\}$ in X^{so} such that $\theta \leq P_n^*(f_\alpha) \uparrow f$ in X^{so}. Since $M_n \cap M_m = \{ \theta \}$ for $m \neq n$, we have for each $m \neq n$,

$$\langle P_m(x), P_n^*(f_\alpha)\rangle = \langle P_n(P_m(x)), f_\alpha\rangle = 0,\ \forall\, x \in X$$

$$\Rightarrow \quad \langle P_m(x), f\rangle = \langle x, P_m^*(f)\rangle = 0,\ \forall\, x \in X,\ \text{as } P_n^*(f_\alpha) \uparrow f$$

$$\Rightarrow \quad P_m^*(f) = \theta \text{ in } X^{so}.$$

Then, following the arguments as in the preceding paragraph, we

obtain $f = P_n^*(f) \in M_n^*$. Consequently, M_n^* is a band of X^{so}.

(ii) In order to prove the disjointness of M_i^* and M_j^* for $i \neq j$, we show that $|P_i^*(f)| \wedge |P_j^*(g)| = \theta$ in X^{so}, for f, g in X^{so}. Let us, therefore, consider an h in X^{so} such that $\theta \leq h \leq |P_i^*(f)| \wedge |P_j^*(g)|$. Then using the same technique as in the proof of (i), we have the implications

$$\theta \leq h \leq |P_i^*(f)| \Rightarrow P_k^*(h) = \theta,\ \forall\ k \neq i \text{ and } h = P_i^*(h);$$
$$\theta \leq h \leq |P_j^*(g)| \Rightarrow P_k^*(h) = \theta,\ \forall\ k \neq j \text{ and } h = P_j^*(h).$$

Hence $h = \theta$, i.e., $|P_i^*(f)| \wedge |P_j^*(g)| = \theta$, for $i \neq j$. As $P_i^*(f)$ and $P_j^*(g)$ are arbitrary elements in M_i^* and M_j^* respectively, $M_i^* \perp M_j^*$ for $i \neq j$. Thus the collection $\{ M_n^* : n \geq 1 \}$ is a family of pairwise disjoint bands of X^{so}.

(iii) For proving the statement (iii), let us consider f in X^{so} such that $f \perp M_n^*$, for each n in $\mathbb{N}$, i.e., $|f| \wedge |P_n^*(g)| = \theta$, for every $g \in X^{so}$ and n in $\mathbb{N}$. So, in particular, we have $|f| \wedge |P_n^*(f)| = \theta$, for every $n \geq 1$. But for $x \in K$, $\theta \leq P_n(x) \leq x$ and so

$$\langle x, P_n^*(|f|)\rangle = \langle P_n(x), |f|\rangle \leq \langle x, |f|\rangle$$
$$\Rightarrow \quad P_n^*(|f|) \leq |f|, \text{ in } X^{so}.$$

Hence $|P_n^*(f)| \leq |f|$ and so $\theta = |f| \wedge |P_n^*(f)| = |P_n^*(f)|$, for each n in $\mathbb{N}$.

Now for $x \in X$, $\sum_{i=1}^{n} P_i(x) \xrightarrow{(o)} x$, as $\{ M_n : n \geq 1 \}$ is an order decomposition for X. Therefore,

$$f(x) = o\text{-}\lim_{k\to\infty} f\Big(\sum_{i=1}^{k} P_i(x)\Big) = o\text{-}\lim_{k\to\infty} \sum_{i=1}^{k} (P_i^*(f))(x) = 0,\ \forall\ x \in X.$$

Thus $f = \theta$ in X^{so}. Hence $\{ M_n^* : n \geq 1\}$ is complete in X^{so}.

(iv) As M_n^* is a band in X^{so} which is an order complete Riesz space by Proposition 1.2.10(i) and Theorem 1.2.20(ii), (iv) follows; cf. the Remark following the Definition 1.2.12. ∎

Combining the statements (i) to (iv) of Proposition 6.2.4, we have the main result of this section contained in

Theorem 6.2.5: If $\{ M_n, P_n \}$ is a countable order decomposition for an order complete Riesz space X, then the sequence $\{ M_n^*, P_n^* \}$ forms an order decomposition for the sequential order dual X^{so} of X such that each f in X^{so} has the representation

$$f = \underset{n}{S}\, P_n^*(f) = o\text{-}\lim_{n\to\infty} \sum_{i=1}^{n} P_i^*(f).$$

Proof: In order to establish that $\{M_n^*, P_n^*\}$ is an order decomposition for X^{so}, it suffices to prove that $P_n^* \equiv P_{M_n^*}$ for $n \geq 1$ in view of Proposition 6.2.4. As

$P_{M_n^*}(f) = \sup \{P_n^*(g):\ g \in X^{so},\ \theta \leq P_n^*(g) \leq f\}$, $f \geq \theta$ in X^{so}, $n \in \mathbb{N}$,

we have to verify the equalities

$P_n^*(f) = \sup \{P_n^*(g): g \in X^{so} \text{and } \theta \leq P_n^*(g) \leq f\}$, $\forall n \in \mathbb{N}$, $f \geq \theta$ in X^{so}.

Since $\theta \leq P_n(x) \leq x$, $\forall\ n \in \mathbb{N}$, $x \in K$, by Proposition 1.2.26(ii), using (6.2.1), we infer that $P_n^*(f) \leq f$, for each n in $\mathbb{N}$ and $f \geq \theta$ in X^{so}. Hence, $P_n^*(f) \leq P_{M_n^*}(f)$, for all $n \in \mathbb{N}$ and $f \geq \theta$ in X^{so}.

For proving that $P_{M_n^*}(f) \leq P_n^*(f)$, for a given n in $\mathbb{N}$ and $f \geq \theta$ in X^{so}, consider any g in X^{so} with $\theta \leq P_n^*(g) \leq f$. Then $\theta \leq P_n^*(P_n^*(g)) \leq P_n^*(f)$, since P_n^* is a positive map from X^{so} to X^{so}. Also, from the relations $P_n(P_n(x)) = P_n(x)$, for $x \in X$ and (6.2.1), we have $P_n^*(P_n^*(g)) = P_n^*(g)$ and so $\theta \leq P_n^*(g) \leq P_n^*(f)$. Therefore $P_{M_n^*}(f) \leq P_n^*(f)$; and consequently, $P_n^* = P_{M_n^*}$, $\forall\ n \in \mathbb{N}$. The last statement of the proposition is an immediate consequence of Theorem 1.2.18 and this completes the proof.■

As a consequence of the above result, we derive the sufficient conditions for the order perfectness of a Riesz space

in the following form

Theorem 6.2.6: Let X be an order complete Riesz space having a countable order decomposition $\{M_n, P_n\}$ such that every $\sigma(X, X^{so})$-bounded set in X is order bounded in X. Then X is order perfect if each M_n is order perfect.

Proof: Regarding the maps P_n from X onto M_n, let us first note from Proposition 6.2.3 that $P_n^*(f) \in X^{so}$ and $P_n^{**}(\phi) \in (M_n^{so})^{so}$, for each f in M_n^{so} and ϕ in $(X^{so})^{so}$.

In order to prove the order perfectness of X, let us now consider an arbitrary element ϕ in $(X^{so})^{so}$. Write $\psi_n = P_n^{**}(\phi)$, $n \geq 1$. Then $\psi_n \in (M_n^{so})^{so}$, $n \geq 1$. Also, from (6.2.2)

(*) $\qquad \psi_n(f) = (P_n^{**}(\phi))(f) = \phi(P_n^*(f)),\ \forall\ f \in M_n^{so}$.

As M_n is order perfect for each $n \geq 1$, there exists a sequence $\{x_n\}$ in X with $x_n \in M_n$, for every n in $\mathbb{N}$, such that

(**) $\qquad \psi_n(g) = g(x_n),\ \forall\ g \in M_n^{so}$.

We now show that the set $B = \{ \sum_{i=1}^{n} x_i : n \in \mathbb{N} \}$ is $\sigma(X, X^{so})$-bounded. Therefore, consider $h \in X^{so}$. As $x_i \in M_i$, $P_i(x_i) = x_i$, $i \geq 1$ and so for $n \geq 1$, we get

$$h(\sum_{i=1}^{n} x_i) = \sum_{i=1}^{n} h(x_i) = \sum_{i=1}^{n} h(P_i(x_i)) = \sum_{i=1}^{n} (P_i^*(h))(x_i) = \sum_{i=1}^{n} g_i(x_i),$$

where $g_i = (P_i^*(h))|_{M_i}$, the restriction of $P_i^*(h)$ on M_i such that $g_i \in M_i^{so}$, for each $i \in \mathbb{N}$. Hence, by (**) and (*), for $n \geq 1$, we get

$$h(\sum_{i=1}^{n} x_i)=\sum_{i=1}^{n} g_i(x_i)=\sum_{i=1}^{n} \psi_i(g_i)=\sum_{i=1}^{n} \phi(P_i^*(g_i))=\phi(\sum_{i=1}^{n} P_i^*(g_i))=\phi(\sum_{i=1}^{n} P_i^*(h)),$$

since

$$\begin{aligned} P_i^*(g_i)(x) = P_i^*((P_i^*(h))|_{M_i})(x) = (P_i^*(h))|_{M_i}(P_i(x)) &= (P_i^*(h))(P_i(x)) \\ &= h(P_i(x)) \\ &= (P_i^*(h))(x),\ x \in X,\ i \in \mathbb{N}. \end{aligned}$$

As $\sum_{i=1}^{n} P_i^*(h) \xrightarrow{(o)} h$, as $n \to \infty$ in X^{so} by Theorem 6.2.5 and $\phi \in (X^{so})^{so}$, $\phi(\sum_{i=1}^{n} P_i^*(h)) = h(\sum_{i=1}^{n} x_i) \longrightarrow \phi(h)$, as $n \to \infty$. Thus B is a $\sigma(X,X^{so})$-bounded and so order bounded set in X by the hypothesis. Consequently, $\{x_n\}$ is an order bounded set in X and hence $\sup_n x_n^+$ and $\sup_n x_n^-$ exist in X by the order completeness of X. If $x = \sup_n x_n^+ - \sup_n x_n^- = Sx_n$, cf. Definition 1.2.15(iii), we have by Proposition 1.2.16(iii)

$$x = \sum_{n=1}^{\infty} x_n = o\text{-}\lim_{n\to\infty} \sum_{i=1}^{n} x_i .$$

Finally, to complete the proof, we show that the element x obtained as above, corresponds to ϕ such that $\phi(f) = f(x)$, $\forall\ f \in X^{so}$. Therefore, consider $f \in X^{so}$. Then

$$f(x) = f(o\text{-}\lim_{n\to\infty} \sum_{i=1}^{n} x_i) = o\text{-}\lim_{n\to\infty} \sum_{i=1}^{n} f(P_i(x_i))$$
$$= o\text{-}\lim_{n\to\infty} \sum_{i=1}^{n} (P_i^*(f))(x_i) = o\text{-}\lim_{n\to\infty} \sum_{i=1}^{n} h_i(x_i),$$

where $h_i = P_i^*(f)|_{M_i} \in M_i^{so}$, $\forall\ i \geq 1$. Thus, by (**) and (*),

$$f(x) = o\text{-}\lim_{n\to\infty} \sum_{i=1}^{n} \phi(P_i^*(h_i)) = o\text{-}\lim_{n\to\infty} \phi(\sum_{i=1}^{n} P_i^*(h_i))$$
$$= o\text{-}\lim_{n\to\infty} \phi(\sum_{i=1}^{n} P_i^*(f)) = \phi(f)$$

as $\sum_{i=1}^{n} P_i^*(f) \xrightarrow{(o)} f$ in X^{so} by Theorem 6.2.5 and $\phi \in (X^{so})^{so}$. Since $f \in X^{so}$ is arbitrary, the result follows.∎

Before we pass on to an example of an order decomposition in a VVSS, let us recall the notation δ_n^x, for $x \in X$ and $n \in \mathbb{N}$, from Section 1.4 and write

$$U_n = \{\ \delta_n^x : x \in X\ \},\ n \geq 1.$$

Also, *define a sequence* $\{R_n\}$ *of linear operators from* $\Omega(X)$ to $\phi(X)$ by

$$R_n(\bar{x}) = \delta_n^{x_n},$$

where $\bar{x} = \{x_n\} \in \Omega(X)$, Then we have

Theorem 6.2.7: Let X be an order complete Riesz space such that $\Lambda(X)$ is an ideal in $\Omega(X)$. Then $\{U_n, R_n\}$ is an order decomposition for $\Lambda(X)$.

Proof: For proving the result, we first show that U_n's are bands in $\Lambda(X)$. Clearly, these are linear subspaces of $\Lambda(X)$. Further, if $\bar{y} \in \Lambda(X)$ is such that $|\bar{y}| \le |\delta_n^x|$ for some x in X and given n in $\mathbb{N}$, then $|y_n| \le x$, and $y_j = \theta$, for $j \ne n$. Thus $\bar{y} = \delta_n^{y_n}$, i.e., $\bar{y} \in U_n$ and so U_n is an ideal of $\Lambda(X)$. For showing that each U_n is order closed, for a given n in $\mathbb{N}$, consider a net $\{x_\alpha : \alpha \in \Lambda\} \subset K$ such that $\bar{\theta} \le \delta_n^{x_\alpha} \uparrow \bar{x}$ in $\Lambda(X)$. Then $\bar{\theta} \le \delta_n^{x_\alpha} \le \delta_n^{x_n}$, $\forall\ \alpha$ in Λ and hence $\bar{x} \le \delta_n^{x_n}$. Thus $\bar{x} = \delta_n^{x_n} \in U_n$ and so U_n is a band for each n in $\mathbb{N}$.

U_n's are pairwise disjoint, as $|\delta_n^x| \wedge |\delta_m^y| = \delta_n^{|x|} \wedge \delta_m^{|y|} = \bar{\theta}$, for $n \ne m$ and for any x,y in X.

For proving the completeness of the sequence $\{U_n\}$ in $\Lambda(X)$, consider $\bar{z} = \{z_n\}$ in $\Lambda(X)$ such that $\bar{z}$ is disjoint from each of these U_n's. Since $\delta_n^{z_n} \in U_n$, for each $n \ge 1$, we get in particular,

$$|\bar{z}| \wedge |\delta_n^{z_n}| = \bar{\theta},\ \forall\ n \ge 1$$

$$\Rightarrow \qquad z_n = \theta,\ \forall\ n \ge 1, \text{ i.e., } \bar{z} = \bar{\theta}.$$

As $\Lambda(X)$ is an ideal in $\Omega(X)$, each U_n is a projection band; cf. Proposition 1.2.10(i) and the Remark following the Definition 1.2.12. Thus $\{U_n : n \ge 1\}$ is a complete family of pairwise disjoint bands in $\Lambda(X)$.

Now, for completing the proof, it remains to show that for $\bar{x} \ge \bar{\theta}$ in $\Lambda(X)$, $R_n(\bar{x}) = P_{U_n}(\bar{x})$, where

$$P_{U_n}(\bar{x}) = \sup \{\bar{y} \in U_n : \bar{\theta} \le \bar{y} \le \bar{x}\}.$$

Clearly, $R_n(\bar{x}) = \delta_n^{x_n} \leq P_{U_n}(\bar{x})$ for $\bar{x} \geq \bar{\theta}$ in $\Lambda(X)$. Conversely, if $\bar{y} \in U_n$ is such that $\bar{\theta} \leq \bar{y} \leq \bar{x}$, then $\bar{y} = \delta_n^{y}$ for some y in X and $\delta_n^{y} \leq \delta_n^{x_n}$. Therefore, $P_{U_n}(\bar{x}) \leq \delta_n^{x_n} = R_n(\bar{x})$. Hence, $\{U_n, R_n\}$ is an order decomposition of $\Lambda(X)$.■

Applying Theorem 6.2.5, we derive from the above result, the following

Corollary 6.2.8: If R_n^* is the adjoint of R_n, for $n \in \mathbb{N}$, defined as in (6.2.1) with P_n, X, X^{so}, being replaced by R_n, $\Lambda(X)$, $[\Lambda(X)]^{so}$, and $U_n^* = R_n^*([\Lambda(X)]^{so})$, then $\{U_n^*, R_n^*\}$ is an order decomposition for $[\Lambda(X)]^{so}$.

Proof: Immediate.■

Remark: Since $[\Lambda(X)]^{so}$ can be identified with $\Lambda^x(X^{so})$ in view of Theorem 3.4.2, and $\Lambda^x(X^{so})$ which is an ideal in $\Omega(X^{so})$ by Theorem 3.4.1, also possesses an order decomposition $\{V_n, Q_n\}$, where $V_n = \{\delta_n^f : f \in X^{so}\}$ and $Q_n(\bar{f}) = \delta_n^{f_n}$, $\bar{f} = \{f_n\} \in \Lambda^x(X^{so})$ by Theorem 6.2.7; it is natural to inquire the relationship of this decomposition with the one obtained in the Corollary 6.2.8. In the sense of identification of Theorem 3.4.2 between $[\Lambda(X)]^{so}$ and $\Lambda^x(X^{so})$, we have the answer contained in

Theorem 6.2.9: The order decomposition $\{U_n^*, R_n^*\}$ is the same as $\{V_n, Q_n\}$.

Proof: Consider an arbitrary element $R_n^*(\bar{f})$ of U_n^* with $\bar{f} \in [\Lambda(X)]^{so}$. Then for $\bar{x} \in \Lambda(X)$, we have

$$(+) \qquad \langle \bar{x}, R_n^*(\bar{f})\rangle = \langle R_n(\bar{x}), \bar{f}\rangle = \langle \delta_n^{x_n}, \bar{f}\rangle = \langle \bar{x}, \delta_n^{f_n}\rangle$$

$$\Rightarrow \qquad R_n^*(\bar{f}) = \delta_n^{f_n} \in V_n.$$

Thus $U_n^* \subset V_n$. For the other inclusion, consider an arbitrary element δ_n^f of V_n where $f \in X^{so}$. Then for $\bar{x} \in \Lambda(X)$,

$$\langle \bar{x}, \delta_n^f\rangle = \langle R_n(\bar{x}), \delta_n^f\rangle = \langle \bar{x}, R_n^*(\delta_n^f)\rangle.$$

$$\Rightarrow \qquad \delta_n^f = R_n^*(\delta_n^f) \in U_n^*.$$

Hence, $U_n^* \equiv V_n$, $\forall\ n\geq 1$.

Also, it follows from (+) that $R_n^*(\bar{f}) = Q_n(\bar{f})$, for any $\bar{f}$ in $[\Lambda(X)]^{so} \equiv \Lambda^x(X^{so})$. Hence $\{U_n^*, R_n^*\}$ is the same as $\{V_n, Q_n\}$ in the sense of identification of Theorem 3.4.2.■

Relating the order perfectness of the space $\Lambda(X)$ with its perfectness, we establish

Theorem 6.2.10: If X is an order perfect, order complete Riesz space and $\Lambda(X)$ is an ideal of $\Omega(X)$, then $\Lambda(X)$ is a perfect VVSS if and only if it is order perfect.

Proof: Let us first assume that $\Lambda(X)$ is perfect and consider a member $\bar{F}$ in $[[\Lambda(X)]^{so}]^{so}$. Since $[[\Lambda(X)^{so}]^{so}$ can be identified with $\Lambda^{xx}[(X^{so})^{so}]$, in view of Theorems 3.4.1, 3.4.2, we can write $\bar{F}=\{F_n\}$, with $F_n \in (X^{so})^{so}$, $n \geq 1$. Now by our hypothesis of order perfectness of X, we can find a sequence $\{x_n\}$ in X such that $F_n(f) = f(x_n)$, for each f in X^{so} and n in $\mathbb{N}$. The order perfectness of $\Lambda(X)$ would follow if we show that $\bar{x} = \{x_n\} \in \Lambda(X)$ and $\bar{J}(\bar{x}) = \bar{F}$, where $\bar{J}$ is the canonical embedding from $\Lambda(X)$ to $[[\Lambda(X)]^{so}]^{so}$.

In order to show that $\bar{x} \in \Lambda(X)$, consider $\bar{f}=\{f_n\}$ in $\Lambda^x(X^{so})$. Then

$$\sum_{n=1}^{\infty} |f_n(x_n)| = \sum_{n=1}^{\infty} |F_n(f_n)| < \infty$$

$$\Rightarrow \qquad \bar{x} \in \Lambda^{xx}(X) = \Lambda(X).$$

Further, $\bar{J}(\bar{x}) = \bar{F}$, since for any $\bar{f} \in \Lambda^x(X^{so})$, we have

$$\bar{J}(\bar{x})(\bar{f}) = \bar{f}(\bar{x}) = \sum_{n=1}^{\infty} f_n(x_n) = \sum_{n=1}^{\infty} F_n(f_n) = \bar{F}(\bar{f}).$$

Hence $\Lambda(X)$ is order perfect.

Now, assume that $\Lambda(X)$ is an order perfect space. For proving that $\Lambda(X)$ is a perfect space, it suffices to prove that $\Lambda^{xx}(X) \subset \Lambda(X)$. Therefore, consider $\bar{x}$ in $\Lambda^{xx}(X)$. If J is the

canonical mapping from X onto $(X^{so})^{so}$, for any $\bar{f}=\{f_n\}$ in $\Lambda^x(X^{so})$, we have

$$\sum_{n=1}^{\infty} |J(x_n)(f_n)| = \sum_{n=1}^{\infty} |f_n(x_n)| < \infty$$

$$\Rightarrow \qquad \{J(x_n)\} \in \Lambda^{xx}[(X^{so})^{so}].$$

As $\Lambda(X)$ is order perfect, there exists an element $\bar{y} = \{y_n\}$ in $\Lambda(X)$ such that $\bar{J}(\bar{y}) = \{J(x_n)\}$. Then, for $\bar{f} \in \Lambda^x(X^{so})$

$$(*) \qquad \bar{J}(\bar{y})(\bar{f}) = \bar{f}(\bar{y}) = \sum_{n=1}^{\infty} f_n(y_n) = \sum_{n=1}^{\infty} J(y_n)(f_n).$$

Since $\sum_{n=1}^{\infty} |f_n(y_n)| < \infty$, we infer that $\{J(y_n)\} \in \Lambda^{xx}[(X^{so})^{so}]$ and thus

$$\{J(y_n)\} = \bar{J}(\bar{y}) = \{J(x_n)\}$$

$$\Rightarrow \qquad J(y_n) = J(x_n),\ \forall\ n\geq 1 \text{ or } x_n=y_n,\ \forall\ n \geq 1, \text{ i.e., } \bar{x} = \bar{y}.$$

Hence $\bar{x} \in \Lambda(X)$ and therefore $\Lambda^{xx}(X) \subset \Lambda(X)$. This completes the proof.■

Remark: Let us remark here that the condition of order perfectness of M_n's in Theorem 6.2.6 is not necessary one in general. For instance, we have

Example 6.2.11: Let X be an order complete Riesz space such that $\Lambda(X)$ is an ideal in $\Omega(X)$, with the countable order decomposition $\{U_n, R_n\}$ as considered in Theorem 6.2.7. Further, assume that X and $\Lambda(X)$ both are order perfect spaces. Now, in the space $\Lambda^{xx}[(X^{so})^{so}]$, consider the subspaces $\{W_n\}$ and the linear operators $\{T_n\}$, $T_n : \Lambda^{xx}[(X^{so})^{so}] \longrightarrow W_n$, defined by

$$W_n = \{\delta_n^F : F \in (X^{so})^{so}\} \text{ and } T_n(\bar{F}) = \delta_n^{F_n},\ n \in \mathbb{N},$$

where $\bar{F}=\{F_n\} \in \Lambda^{xx}[(X^{so})^{so}]$. Making use of Theorem 6.2.7, $\{W_n, T_n\}$ is an order decomposition for $\Lambda^{xx}[(X^{so})^{so}]$. If we consider the map $S_n : U_n \longrightarrow W_n$, defined by

$$S_n(\delta_n^x) = \delta_n^{J(x)},\ x \in X,$$

then S_n is a Riesz isomorphism from U_n onto W_n, $n \geq 1$. However, if we regard U_n, themselves as OVVSS over the space X, then we can easily establish that

$$[U_n(X)]^{so} \equiv U_n^x(X^{so}) = \Omega(X^{so});$$

and

$$[\Omega(X^{so})]^{so} \equiv \phi\ [(X^{so})^{so}].$$

Since W_n's are proper subspaces of $\phi\ [(X^{so})^{so}]$ and $\bar{J}(U_n) = W_n$, $\forall$ $n \geq 1$, U_n's are not order perfect.

As a particular example, we may consider the space ℓ^∞ of all real bounded sequences, which is order perfect by the Note 1.4.8. Also, every $\sigma(\ell^\infty, \ell^1)$ - bounded set in ℓ^∞ is bounded by a point in ℓ^∞; cf. [96], p.107. However, $U_n = \{\ \delta_n^a : a \in \mathbb{R}\} = W_n$, $\forall$ $n \in \mathbb{N}$; $U_n^{so} = w$ and $w^{so} = \phi$, $\forall$ $n \in \mathbb{N}$.

6.3 <u>Associated o-Sequence Spaces</u>: As mentioned earlier, in this section we define OVVSS with the help of the projection operators P_n's and P_n^*'s which correspond to a countable order decomposition $\{M_n, P_n\}$ in an order complete Riesz space X. Indeed, if $\{M_n, P_n\}$ is a countable order decomposition for X, then $\{M_n^*, P_n^*\}$ is an order decomposition for X^{so} by Theorem 6.2.5. We now introduce

(6.3.1) $\Lambda_a(X) = \{\ \bar{x} = \{x_n\} : x_n \in M_n,\ n \geq 1$ and $\sum_{i=1}^{n} x_i$ order converges in X$\}$

and

(6.3.2) $\Delta_a(X^{so}) = \{\bar{f} = \{f_n\} : f_n \in M_n^*,\ n \geq 1$ and $\sum_{i=1}^{n} f_i$ order converges in $X^{so}\}$

In terms of the projection operators P_n and P_n^*, we have the following representation of these spaces:

Proposition 6.3.3: (i) $\Lambda_a(X) = \{\bar{x} : \bar{x} = \{P_n(x)\},\ x \in X\}$; and

(ii) $\Delta_a(X^{so}) = \{\bar{f} : \bar{f} = \{P_n^*(f)\},\ f \in X^{so}\}$.

Proof: (i) For $\bar{x} \in \Lambda_a(X)$, $\sum_{i=1}^{n} x_i \xrightarrow{(o)} x$, for some x in X. As P_n's are

sequentially order continuous and $M_n \perp M_m$, $m \neq n$, we have $x_n = P_n(x)$, for each n in $\mathbb{N}$. Thus $\Lambda_a(X) \subset \{ \bar{x} : \bar{x} = \{P_n(x)\}, x \in X\}$. The other inclusion follows from Theorem 1.2.18.

(ii) The proof is analogous to that of (i) and so omitted.■

Clearly, $\Lambda_a(X)$ and $\Delta_a(X^{so})$ are subspaces of $\Omega(X)$ and $\Omega(X^{so})$, which may not contain $\phi(X)$ and $\phi(X^{so})$ respectively. But for the subspace ordering, we have

Proposition 6.3.4: $\Lambda_a(X)$ and $\Delta_a(X^{so})$ are ideals of $\Omega(X)$ and $\Omega(X^{so})$ respectively.

Proof: We prove the result for $\Lambda_a(X)$ as the result for $\Delta_a(X^{so})$ would follow on similar lines.

In order to show that $\Lambda_a(X)$ is a Riesz subspace of $\Omega(X)$, consider an element $\bar{x} = \{x_n\}$ of $\Lambda_a(X)$ and assume that x is the order limit of the sequence $\{ \sum_{i=1}^{n} x_i\}$. As M_n's are pairwise disjoint and $x_n \in M_n$, for $n \geq 1$, we have by Proposition 1.2.14

$$(+) \qquad \sum_{i=1}^{n} |x_i| = | \sum_{i=1}^{n} x_i |, \ \forall\ n \in \mathbb{N}$$

$$\Rightarrow \qquad \sum_{i=1}^{n} |x_i| \xrightarrow{(o)} |x|, \text{ as } n \longrightarrow \infty.$$

Thus $|\bar{x}| = \{|x_n|\} \in \Lambda_a(X)$, as $|x_n| \in M_n$, $\forall\ n \geq 1$.

For proving the ideal character of $\Lambda_a(X)$ in $\Omega(X)$, consider an element $\bar{y}=\{y_n\}$ in $\Omega(X)$ such that $\bar{\theta} \leq \bar{y} \leq \bar{x}$, for some $\bar{x} = \{x_n\} \in \Lambda_a(X)$. Then, $\theta \leq y_i \leq x_i$, $\forall\ i \geq 1$ and so $\{ \sum_{i=1}^{n} y_i\}$ is an increasing sequence in X. As M_i's are ideals in X, $y_i \in M_i$, $\forall\ i \geq 1$. Further, the sequence $\{\sum_{i=1}^{n} y_i\}$, being an order bounded increasing sequence, order converges to its supremum which exists as X is order complete. Hence $\bar{y}=\{y_n\} \in \Lambda_a(X)$. Consequently, $\Lambda_a(X)$ is a Riesz subspace of $\Omega(X)$ with the property that $\bar{\theta} \leq \bar{y} \leq \bar{x}$, $\bar{x} \in \Lambda_a(X)$, yields $\bar{y} \in \Lambda_a(X)$. Hence it is an ideal of $\Omega(X)$.■

Concerning the duality of $\Lambda_a(X)$ and $\Delta_a(X^{so})$, we prove

Proposition 6.3.5: The spaces $\Lambda_a(X)$ and $\Delta_a(X^{so})$ form a dual pair for the bilinear form

$$(*) \qquad B(\bar{x},\bar{f}) = \sum_{i=1}^{\infty} \langle P_i(x), P_i^*(f)\rangle,$$

where $\bar{x} = \{P_i(x)\}$, $\bar{f} = \{P_i^*(f)\}$, for $x \in X$ and $f \in X^{so}$.

Proof: Let us first note that the series in (*) is a convergent series, as for $x \in X$, $\sum_{i=1}^{n} P_i(x) \xrightarrow{(o)} x$, as $n \longrightarrow \infty$, by Theorem 1.2.18, and so for any f in X^{so},

$$\sum_{i=1}^{n} \langle P_i(x), P_i^*(f)\rangle = \sum_{i=1}^{n} \langle P_i(x), f\rangle$$
$$\longrightarrow f(x), \text{ as } n \longrightarrow \infty.$$

Now, if $B(\bar{x},\bar{f}) = 0$, for all $\bar{f}$ in $\Delta_a(X^{so})$, then $\bar{x} = \bar{\theta}$; for otherwise, $P_i(x) \neq \theta$ for some i in $\mathbb{N}$ and x in X, would yield that $B(\bar{x}, \delta_i^{P_i^*(f)}) = \langle P_i(x), P_i^*(f)\rangle \neq 0$, for some $f \in X^{so}$, by our basic assumption for the dual pair $\langle X, X^{so}\rangle$. Similarly, we can establish that $B(\bar{x},\bar{f}) = 0$, $\forall$ $\bar{x}$ in $\Lambda_a(X)$, implies that $\bar{f} = \bar{\theta}$. Thus $\langle \Lambda_a(X), \Delta_a(X^{so})\rangle$ is a dual pair.■

Next, we establish

Proposition 6.3.6: With X, X^{so}, $\Lambda_a(X)$, $\Delta_a(X^{so})$ as mentioned above, the spaces X and X^{so} are respectively Riesz isomorphic to $\Lambda_a(X)$ and $\Delta_a(X^{so})$.

Proof: Consider the mappings π_1: $\Lambda_a(X) \longrightarrow X$ and $\pi_2: X^{so} \longrightarrow \Delta_a(X^{so})$ defined by

$$\pi_1(\{P_n(x)\}) = x, \; x \in X \quad ; \quad \pi_2(f) = (\{P_n^*(f)\}), \; f \in X^{so}.$$

Clearly, π_1 and π_2 are one to one linear mappings from $\Lambda_a(X)$ onto X and X^{so} onto $\Delta_a(X^{so})$ respectively. Also, by Proposition 1.2.26, π_1 and π_2 are Riesz homomorphisms, for

$$|\pi_1(\{P_n(x)\})| = |x| = \pi_1(\{P_n(|x|)\}) = \pi_1(\{|P_n(x)|\}) = \pi_1(|\{P_n(x)\}|);$$

and

$|\pi_2(\{P_n^*(f)\})| = |f| = \pi_2(\{P_n^*(|f|)\}) = \pi_2(\{|P_n^*(f)|\}) = \pi_2(|\{P_n^*(f)\}|)$.
Hence X and X^{so} are Riesz isomorphic to $\Lambda_a(X)$ and $\Delta_a(X^{so})$ under π_1 and π_2 respectively.■

Recalling the spaces $\ell^1(X)$ and $\ell^1(X^{so})$ defined corresponding to an order complete Riesz space X and its sequential order dual X^{so} from (3.4.4), and taking into account the intrinsic properties of the spaces $\Lambda_a(X)$ and $\Delta_a(X^{so})$ established above, we prove

Proposition 6.3.7: $\Lambda_a(X)$ and $\Delta_a(X^{so})$ are bands in $\ell^1(X)$ and $\ell^1(X^{so})$ respectively.

Proof: Invoking the proof of Proposition 6.3.4, we observe that the spaces $\Lambda_a(X)$ and $\Delta_a(X^{so})$ are ideals of $\ell^1(X)$ and $\ell^1(X^{so})$ respectively.

For showing that $\Lambda_a(X)$ is order closed in $\ell^1(X)$, consider a net $\{\bar{x}^\alpha : \alpha \in \Lambda\}$ in $\Lambda_a(X)$ such that $\bar{\theta} \le \bar{x}^\alpha \uparrow \bar{x}$ in $\ell^1(X)$. Then $\theta \le x_i^\alpha \uparrow x_i$ in X, for each $i \ge 1$, by Proposition 3.2.7(ii). But $\{x_i^\alpha : \alpha \in \Lambda\} \subset M_i$ and M_i is a band in X. Therefore, $x_i \in M_i$ for each $i \ge 1$ and consequently, $\bar{x} \in \Lambda_a(X)$. Hence $\Lambda_a(X)$ is a band of $\ell^1(X)$. Similarly, we can prove that $\Delta_a(X^{so})$ is a band of $\ell^1(X^{so})$.■

6.4 <u>o-Similarity</u>: In this final section, we study the o-similarity between countable order decompositions of Riesz spaces, introduced with the help of order linear mappings, as follows:

Definition 6.4.1: Let X and Y be two order complete Riesz spaces with countable order decompositions $\{M_n, P_n\}$ and $\{N_n, Q_n\}$. Then $\{M_n, P_n\}$ and $\{N_n, Q_n\}$ are said to be *o-similar* if there exist Riesz isomorphisms T_n from M_n to N_n, such that T_n and T_n^{-1} are

sequentially order continuous from M_n onto N_n and N_n onto M_n respectively, and

(i) $\sum_{i=1}^{n} x_i$, $x_i \in M_i$,order converges in X $\Rightarrow$

$$\sum_{i=1}^{n} T_i(x_i) \text{ order converges in } Y;$$

(ii) $\sum_{i=1}^{n} y_i$, $y_i \in N_i$,order converges in Y $\Rightarrow$

$$\sum_{i=1}^{n} T_i^{-1}(y_i) \text{ order converges in } X.$$

A characterization of o-similar order decompositions is given in

Proposition 6.4.2: Let X, Y, $\{M_n,P_n\}$ and $\{N_n,Q_n\}$ be as above. Then $\{M_n,P_n\}$ and $\{N_n,Q_n\}$ are o-similar if and only if there exists a σ- Riesz isomorphism A from X onto Y with A^{-1} sequentially order continuous such that $A(M_n)=N_n$, for each n in $\mathbb{N}$.

Proof: Let us first assume that $\{M_n,P_n\}$ and $\{N_n,Q_n\}$ are o-similar. Then for each n≥1, there exist Riesz isomorphisms $T_n:M_n \longrightarrow N_n$, satisfying the properties stated in Definition 6.4.1. For n ≥ 1, define $A_n:X \longrightarrow Y$ by

$$A_n(x) = \sum_{i=1}^{n} T_i(P_i(x)), \; x \in X.$$

A_n's are positive linear maps from X to Y such that for $x \in M_i$, $A_n(x) = \theta$, for n <i and $A_n(x) = T_i(x)$, for n ≥ i, where i,n ∈ $\mathbb{N}$. Each A_n is sequentially order continuous, for if $\{x_k\}$ is a sequence in X with $x_k \downarrow \theta$ in X, then $P_i(x_k) \downarrow \theta$ in X. As M_i's are ideals in X, $P_i(x_k) \downarrow \theta$ in M_i and so $T_i(P_i(x_k)) \underset{k}{\downarrow} \theta$ in Y, ∀ i≥1. Therefore, $A_n(x_k) \underset{k}{\downarrow} \theta$ in Y, for each n in $\mathbb{N}$. Hence by Theorem 1.2.20(iii), A_n's are sequentially order continuous linear maps.

Now define $A: X \longrightarrow Y$, by

$$A(x)=\text{o-}\lim_{n\to\infty} A_n(x),\ x \in X.$$

Let us note that A is a well defined map from X to Y, since for $x \in X$, $\sum_{i=1}^{n} P_i(x) \xrightarrow{(o)} x$ in X and so by Definition 6.4.1(i), $A_n(x)=\sum_{i=1}^{n} T_i(P_i(x))$ order converges in Y. Further, A is clearly a linear map. It is one to one; for if $A(x) = \theta$, then $\{\sum_{i=1}^{n} T_i(P_i(x))\}$ order converges to θ in Y. But $\sum_{i=1}^{n} |T_i(P_i(x))| = |\sum_{i=1}^{n} T_i(P_i(x))|$, for each $n \in \mathbb{N}$, as $N_i \perp N_j$, $i \neq j$. Therefore, $|T_i(P_i(x))| = \theta$, $\forall i \geq 1$. Since T_i's are one to one maps, $P_i(x)=\theta$, $\forall i \geq 1$. Hence $x = \theta$.

In order to show that A is onto, consider $y \in Y$. As T_n's are onto, we can find a sequence $\{x_n\}$ in X with $x_n \in M_n$, $\forall n \geq 1$, such that

$$Q_n(y) = T_n(x_n), \text{ i.e., } x_n = T_n^{-1}(Q_n(y)),\ \forall n \geq 1.$$

Hence by Definition 6.4.1(ii), $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} T_i^{-1}(Q_i(y)) \xrightarrow{(o)} x$, for some x in X, as $\sum_{i=1}^{n} Q_i(y) \xrightarrow{(o)} y$ in Y. We now show that $y = A(x)$. Indeed, it follows from the equalities

$$A(x) = \text{o-}\lim_{n\to\infty} \sum_{i=1}^{n} T_i(P_i(x)) = \text{o-}\lim_{n\to\infty} \sum_{i=1}^{n} T_i(x_i) = \text{o-}\lim_{n\to\infty} \sum_{i=1}^{n} Q_i(y) = y.$$

Also, A is a Riesz homomorphism , since for $x \in X$,

$$\begin{aligned}
|A(x)| &= \text{o-}\lim_{n\to\infty} |A_n(x)| \quad \text{by Proposition 1.2.7(i)} \\
&= \text{o-}\lim_{n\to\infty} \sum_{i=1}^{n} |T_i(P_i(x))| \quad \text{since } N_i \perp N_j,\ \forall\, i \neq j \\
&= \text{o-}\lim_{n\to\infty} \sum_{i=1}^{n} T_i(P_i(|x|)) \quad = A(|x|),
\end{aligned}$$

as $T_i \circ P_i$ is a Riesz homomorphism from X to N_i, for each $i \geq 1$.

For sequential order continuity of A, observe that $\theta \leq A_n \uparrow A$ in $\mathcal{L}^b(X,Y)$ by Theorem 1.2.22(ii). But $\{A_n : n \geq 1\} \subset$

$\mathcal{L}^{so}(X,Y)$ which is a band in $\mathcal{L}^{b}(X,Y)$; hence $A \in \mathcal{L}^{so}(X,Y)$.

Coming to the discussion of A^{-1}, observe that

$$A^{-1}(y) = o\text{-}\lim_{n\to\infty} \sum_{i=1}^{n} T_i^{-1}(Q_i(y)),\ y \in Y;$$

for if $x \in X$ is such that $x = o\text{-}\lim_{n\to\infty} \sum_{i=1}^{n} T_i^{-1}(Q_i(y))$, then $T_i(P_i(x)) = Q_i(y)$, $\forall\ i \geq 1$. As the expression for A^{-1} is similar to that of A, its sequential order continuity follows on the lines of proof for A.

To establish that $A(M_n) = N_n$, $n \in \mathbb{N}$, consider $x \in M_n$. Then, $A(x) = A_n(x) = T_n(x)$. As $T_n(x) \in N_n$, $A(M_n) \subset N_n$. For the other inclusion, if $y \in N_n$, then $y = T_n(x)$, for some $x \in M_n$, as T_n is onto. But in this case, $T_n(x) = A_n(x) = A(x)$. Hence $y \in A(M_n)$ and so $N_n \subset A(M_n)$.

For the converse implication of the result, for each $n \in \mathbb{N}$, define $T_n: M_n \longrightarrow N_n$ as $T_n = A|_{M_n}$, restriction of A on M_n. Then, we can easily establish from the properties of A and A^{-1}, that $\{T_n\}$ is the required sequence of operators, which makes the two order decompositions $\{M_n, P_n\}$ and $\{N_n, Q_n\}$ o-similar.■

In terms of associated o-sequence spaces $\Lambda_a(X)$ and $\Lambda_a(Y)$, we have another characterization of o-similarity contained in

Theorem 6.4.3: Let X, Y, $\{M_n, P_n\}$ and $\{N_n, Q_n\}$ be as above. Then $\{M_n, P_n\}$ and $\{N_n, Q_n\}$ are o-similar if and only if there exists a σ-Riesz isomorphism T from $\Lambda_a(X)$ onto $\Lambda_a(Y)$ with sequentially order continuous T^{-1} such that $T(U_i) = V_i$, where $U_i = \{\delta_i^x : x \in M_i\}$ and $V_i = \{\delta_i^y : y \in N_i\}$, for $i \geq 1$.

Proof: For proving the necessity, we have by the hypothesis and Proposition 6.4.2, a Riesz isomorphism A from X onto Y such that

A and A^{-1} are sequentially order continuous and $A(M_n) = N_n$, $n \geq 1$. Hence, for $\bar{x} = \{x_n\} \in \Lambda_a(X)$, $\{A(x_n)\} \in \Lambda_a(Y)$. Thus we can define a linear map T from $\Lambda_a(X)$ to $\Lambda_a(Y)$ by

$$T(\bar{x}) = \{A(x_n)\}, \text{ for } \bar{x} = \{x_n\} \in \Lambda_a(X).$$

As A is one-one map, T is so. Also, T is onto; for if $\bar{y}=\{y_n\}\in \Lambda_a(Y)$, then $y_n= A(x_n)$, for some $x_n \in M_n$, $n \geq 1$, as $A(M_n) = N_n$. Since A^{-1} is sequentially order continuous, $\bar{x}=\{x_n\} \in \Lambda_a(X)$. Further, $T(\bar{x}) = \bar{y}$.

Since $A(M_i) = N_i$, $T(U_i) = V_i$, for each $i\geq 1$. Further, the inverse of T is given by

$$T^{-1}(\{y_i\}) = \{A^{-1}(y_i)\}, \ \bar{y} = \{y_i\} \in \Lambda_a(Y).$$

As A is a Riesz homomorphism, we have the equalities

$$|T(\bar{x})|=|\{A(x_n)\}|=\{|A(x_n)|\}=\{A(|x_n|)\}=T(|\bar{x}|), \ \forall \ \bar{x} = \{x_n\} \in \Lambda_a(X).$$

Thus T is a Riesz homomorphism from $\Lambda_a(X)$ onto $\Lambda_a(Y)$.

In order to show that T is sequentially order continuous, consider a sequence $\{\bar{x}^m\}$ in $\Lambda_a(X)$ such that $\bar{x}^m \downarrow \bar{\theta}$ in $\Lambda_a(X)$, where $\bar{x}^m=\{x_i^m\}$. Then $x_i^m \downarrow_m \theta$ in X, for each $i\geq 1$; indeed, if there is an element a in X such that $\theta \leq a \leq x_i^m$, for each $m \geq 1$, then $a \in M_i$ and so $\delta_i^a\in \Lambda_a(X)$ is such that $\delta_i^a \leq \bar{x}^m$, $\forall$ $m\geq 1$, which yield that $a = \theta$. Consequently, $A(x_i^m) \downarrow_m \theta$ in Y, $\forall$ $i \geq 1$ and hence it can be easily seen that $T(\bar{x}^m) = \{A(x_i^m)\} \downarrow \bar{\theta}$ in $\Lambda_a(Y)$. As T is positive, T is sequentially order continuous.

Similar arguments hold for proving the sequential order continuity of T^{-1}.

In order to establish the sufficiency, for $n \in \mathbb{N}$, define $T_n:M_n\longrightarrow N_n$ by $T_n(x)=y$, where $x \in M_n$, and $y \in N_n$ is such that $T(\delta_n^x) = \delta_n^y$, for $x \in M_n$. As T is a one to one, onto linear mapping from $\Lambda_a(X)$ to $\Lambda_a(Y)$ with $T(U_i) = V_i$, T_n is clearly so

from M_n onto N_n for each n in $\mathbb{N}$. Further, each T_n is a Riesz homomorphism as for $T(\delta_n^x) = \delta_n^y$, $x \in M_n$, $y \in N_n$, we have

$$\delta_n^{|y|} = |\delta_n^y| = |T(\delta_n^x)| = T(\delta_n^{|x|})$$

$$\Rightarrow \qquad |T_n(x)| = |y| = T_n(|x|).$$

Next, we show that T_n's are sequentially order continuous. For fixed $n \in \mathbb{N}$, consider a sequence $\{x_k\} \subset M_n$ such that $x_k \downarrow \theta$ in M_n. As M_n is a regular Riesz subspace of X, $x_k \downarrow \theta$ in X. Hence $\delta_n^{x}k \downarrow \bar{\theta}$ in $\Lambda_a(X)$ and so $T(\delta_n^{x}k) \downarrow \bar{\theta}$ in $\Lambda_a(Y)$. If the sequence $\{y_k\} \subset N_n$ is such that $T(\delta_n^{x}k) = \delta_n^{y}k$, $\forall\ k \geq 1$, it follows clearly from $T(\delta_n^{x}k) \underset{k}{\downarrow} \bar{\theta}$, that $T_n(x_k) = y_k \underset{k}{\downarrow} \theta$ in N_n. As T_n is positive, it is sequentially order continuous.

For proving the sequential order continuity of T_n^{-1}, we make use of sequential order continuity of T^{-1} and proceed on similar lines as in the preceding paragraph.

Finally, we show that the conditions (i), (ii) of Definition 6.4.1, are satisfied by the sequences $\{T_n\}$ and $\{T_n^{-1}\}$. For proving (i) of Definition 6.4.1, consider a sequence $\{x_n\} \subset X$ with $x_i \in M_i$, $\forall\ i \geq 1$ such that $\sum_{i=1}^{n} x_i \xrightarrow{(o)} x$ in X. If $\bar{x} = \{x_n\} \in \Lambda_a(X)$, then $\bar{x}^{(n)} \xrightarrow{(o)} \bar{x}$ in $\Lambda_a(X)$, cf. Proposition 3.2.7(i). If $z_i \in N_i$ is such that $T(\delta_i^{x}i) = \delta_i^{z}i$, for each $i \geq 1$, then by the sequential order continuity of T

$$T(\bar{x}^n) = \sum_{i=1}^{n} \delta_i^{z}i \xrightarrow{(o)} T(\bar{x}) \text{ in } \Lambda_a(Y)$$

$$\Rightarrow \qquad T(\bar{x}^n) = \sum_{i=1}^{n} \delta_i^{z}i \xrightarrow{(o)} T(\bar{x}) \text{ in } \Omega(Y),$$

as $\Lambda_a(Y)$ is a regular subspace of $\Omega(Y)$. But from Proposition 3.2.7(i),

$$T(\bar{x}^n) = \sum_{i=1}^{n} \delta_i^{z}i \xrightarrow{(o)} \bar{z} \quad \text{in } \Omega(Y),$$

where $\bar{z} = \{z_i\}$. Hence $T(\bar{x}) = \bar{z}$ by the uniqueness of o-limit.

Consequently, $\bar{z} \in \Lambda_a(Y)$; in other words, we have the order convergence of the sequence $\{\sum_{i=1}^{n} T_i(x_i)\}$ in Y. Similarly, the condition (ii) of Definition 6.4.1, can be disposed off by using the sequential order continuity of T^{-1} . This establishes the result completely.■

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